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Phil. Trans. R. Soc. Lond. A 1975 **279**, 487-545
doi: 10.1098/rsta.1975.0085

THE INTRINSIC PROPERTIES OF RANK AND NULLITY OF THE LAGRANGE BRACKET IN THE ONE DIMENSIONAL CALCULUS OF VARIATIONS

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(*Communicated by C. T. C. Wall, F.R.S. – Received* 17 *December* 1973 – *Revised* 6 *March* 1975)

CONTENTS

This paper establishes the existence of symplectic structure in *degenerate* variational problems, i.e. problems whose full development involves a hierarchy of equations of constraint as well as various equations of motion.

Any variational problem, degenerate or otherwise, may be called *regular* if the equations of the second variation provide a complete description of the infinitesimal relationships subsisting between any orbit and all its infinitesimal neighbour orbits. It is proved that Poincaré's conserved antisymmetric derived bilinear differential form in the orbit manifold of any regular degenerate problem admits no null vectors other than those which represent infinitesimal deviations due to indeterminacy in the evolution of the orbit. Conversely, it is shown how, given any continuous system of orbits endowed with a conserved antisymmetric closed bilinear differential form having this unique property of rank and nullity, one can construct at least one regular variational

principle, reproducing all the given orbits and no others, and reproducing the given bilinear form. The problem of constructing a non-redundant and canonical set of co-ordinates in the orbit manifold is also treated, and with it is given a particularly simple geometric solution to the problem of reduction of a Pfaffian linear differential form on a manifold.

All theorems and results are formulated and established both for ordinary systems, and for abstract systems in which the basic dependent variables mutually anticommute.

1. Introduction

Since the work of Poincaré (1899) and Cartan (1922) it has been known that Hamilton's variational principle in ordinary Lagrangian mechanics leads to a coordinate-invariant antisymmetric derived bilinear differential form of maximal rank, which by virtue of its property of conservation bestows a time-independent symplectic geometry upon the set of all orbits (Whittaker 1937; Lanczos 1949; Pars 1965; Abraham 1967). In more recent times various developments in relativistic physics have led to the recognition of a large family of variational problems of a less straightforward type (Dirac 1950); problems which are *degenerate* in the sense that the Euler–Lagrange equations lead to non-trivial hierarchic systems of constraints (for which see §2 herein). Sometimes the degeneracy is accompanied by indeterminacy in the dynamic evolution of the orbits. Although there is no difficulty in extending Poincaré's differential form to degenerate cases, its rank can no longer be determined by inspection. The universal rank theorems of the present paper have been developed with precisely this consideration in mind. An extensive account of the invariant and time-independent symplectic structure implied by the present work may be found elsewhere (Allcock 1975).

The investigations of the following pages are concerned with general mathematical attributes of the calculus of variations. Nevertheless it is convenient to retain some terminology from mechanics, since this discipline offers itself as a familiar vehicle ready made to carry the various concepts which are required. In particular we refer to the independent variable t as the *time*, to the stationary curves of the variational principle as *orbits*, to the evolution of the orbit parameters along an orbit as *motion*, and to the basic bilinear form as the *Lagrange bracket*.

The Lagrange bracket (§6) is a concept intrinsic to the orbit system. It is defined on any point of any orbit, and the two differential increments which appear in it represent two infinitesimal displacements of the various orbit parameters from that point to points lying on any two other orbits in the infinitesimal neighbourhood of the first. Thus the dimension of the local tangent space generated by all the admissible orbit differentials of the Lagrange bracket bears no direct relation to the number of coordinates q^α appearing in the action principle, and may be larger, smaller, or equal, depending on how many parameters are needed for the full specification of an orbit. Irrespective of such details of the case, and subject only to a certain requirement of *regularity*, and to the usual assumptions of differentiability and continuity, it will be proved that the rank of the basic bilinear differential form is equal to the dimension of the above-mentioned tangent space if the equations of motion are fully deterministic, and is otherwise smaller. It will be proved also that in the latter case every null vector of the bilinear form represents some possible infinitesimal orbit deviation, which might be realized instantaneously by virtue of the indeterminacy present in the equations of motion at the instant considered or, failing that, by virtue of the cumulative effect of such indeterminacy acting over an extended period of time prior to or subsequent to the instant to which the null vector pertains.

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The converse of the last proposition will also be proved. It will be shown too that if there are any null vectors they generate integral submanifolds in the space of the orbit parameters, the so-called gauge manifolds of electromagnetism and other partially indeterministic physical systems. Full statements of these properties may be found in §§ 7–9.

The proof of the theorems whose import has just been described depends in an essential and irreducible way upon a technique first formulated by Jacobi, though in its main respects anticipated by Lagrange's earlier method of variation of constants. In the Jacobi method one studies the temporal evolution of one orbit relative to an infinitesimally distant neighbour orbit by using the equations of motion once differentiated with respect to whatever parameters are relevant. The equations descriptive of this type of comparison have often been called the variational equations (Whittaker 1937), but in the present context such terminology would be most confusing, and therefore, following an already existing precedent (Pars 1965), we shall call them the Jacobi equations. It is at once evident that every infinitesimal orbit displacement must obey the Jacobi equations; what is not so apparent, yet is in fact quite trivial to demonstrate, is that for general action principles the converse need not be true. Thus we are compelled to distinguish between cases where the action principle is *regular*, and where every solution of the associated Jacobi equations represents a genuine infinitesimal orbit displacement, and cases where the action principle is *singular*, and where the associated Jacobi equations have spurious solutions. The theorems described above apply universally to all regular action principles, irrespective of degeneracy type. For singular action principles our method of proof and also the theorems themselves both break down.

From the above discussion it will be realized that there is a very close connexion indeed between the property of regularity or singularity of a variational principle and the properties of rank and nullity of the associated Lagrange bracket system. This suggests the possibility of a converse theorem. Suppose we are given a differentiable continuum of orbits endowed with a Lagrange bracket having the properties of rank and nullity appropriate to a regular Lagrange function. Can we then with their aid construct a regular Lagrange function such that, on applying to it the standard procedures, we recover from it not only the given orbits, but also the given Lagrange brackets? To this question our answer is unequivocally in the affirmative, and in fact in § 11 a suitable construction for one such Lagrange function will be given explicitly. But there are other solutions besides, in which the number of independent q^α to be varied is different, and therefore the discussion which we are able to give by no means exhausts this interesting converse problem.

Nevertheless, with the established converse theorem in mind, it is certainly no longer legitimate to entertain that narrow yet historically sanctioned point of view once so nicely expressed by Eddington (1952), who quipped that 'from its first introduction, action has always been looked upon as something whose sole *raison d'être* is to be varied – and, moreover, *varied in such a way as to defy the laws of nature*!' Clearly, there is more to stationary principles than this, and whether one adopts the point of view of a mathematician or a physicist it is evident that the proper role of a regular Lagrange function is to serve as combined generating function both for a family of orbits and for an associated symplectic geometry in orbit space.

The various theorems whose import we have now briefly outlined are proved here not only for general systems of the classical type, but also for 'anticlassical' systems, in which the dependent variables mutually anticommute. This mathematical extension of the usual context is valuable in the treatment of quantum systems of the Fermi–Dirac type (Allcock 1975).

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The orbits of a general variational principle may always be viewed as a set of t-dependent points scattered over a t-dependent differentiable manifold whose description, furnished in all that is essential by the variational principle itself, is accomplished globally with the help of redundant but single-valued variables. The great advantages to be gained by working with redundant variables have long been recognized, and in modern relativistic physics the method is well-nigh indispensable. A beautiful and much earlier example is provided by Euler's globally continuous and single-valued spinorial or quaternionic parametrization of the orientation of a rigid body which, besides having the desirable features indicated, enjoys also a composition rule of remarkable elegance and simplicity (see, for example, Whittaker 1937). The main purpose of the present paper is therefore to establish the relevant intrinsic geometric properties of the orbit manifold, and to present them in a generally covariant manner, without recourse to special or locally adapted non-redundant coordinate systems or coordinate patchings. It is however useful at times to set up a non-redundant but canonical system of coordinates on the orbit manifold, and so to obtain an alternative mode of description, in which the Euler–Lagrange equations and the Lagrange bracket appear in Hamilton's canonical form. Section 12 of the paper provides a constructive demonstration that a globally continuous (but in general multi-valued) canonical coordinate system can be established for any regular action principle. The problem here is essentially that of Pfaff. Our resolution of it is more simple and geometric than most, and is novel in that the treatment covers cases where the basic variables mutually anti-commute. The canonical equations themselves are derived in §13, for the most general time-dependent classical or anticlassical Lagrangian system.

For details of the contents of those main sections not mentioned above the reader is referred to the contents list. Each main section is freely subdivided to ensure that every new concept is introduced under a suitable heading, and key results are italicized for easy reference.

2. The variational principle and its constraint chains

Classical variational problems

The class of variational problems to be discussed in this paper is that characterized by a single variational equation

$$\delta \mathbf{A}_1 = 0, \tag{1}$$

where $\mathbf{A}_1$, the Hamilton–Ostrogradski action in mechanics (Whittaker 1937), is a real functional over some interval $[t_1, t_2]$, depending on a set of real functions $q^\alpha(t)$ $(\alpha = 1, 2, \ldots, \nu)$ through a formula of the type

$$\mathbf{A}_1 = \int_{t_1}^{t_2} L_1(t, q^\alpha(t), \mathrm{D}q^\alpha(t), \ldots, \mathrm{D}^N q^\alpha(t))\, \mathrm{d}t, \tag{2}$$

where $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}t$.

The $q^\alpha(t)$ may be ordinary arithmetical variables and if so we shall demand for them, as for the real function L_1 in which they appear, only such properties of differentiability to finite order (with continuity of the last derivative) as are needed to make sense of our various operations. The temporal derivative operation D may if desired be understood in a broad sense, as in the theory of distributions, without prejudice to the validity of the results. But all the other derivatives which are needed (such as $\partial L_1/\partial q^\alpha$) must exist in the narrow sense of traditional calculus.

Anticlassical variational problems

An alternative type of action principle (1), prompted by the presence in nature of Fermi–Dirac systems, is that where the basic variables $q^\alpha(t)$ in (2) are abstract and symbolic objects which obey a universal rule of anticommutation

$$q^\alpha(t)\, q^{\alpha'}(t') = -q^{\alpha'}(t')\, q^\alpha(t) \quad (\forall\, \alpha, \alpha', t, t'), \tag{3}$$

and which possess the reality properties

$$q^\alpha(t)^* = q^\alpha(t), \{q^\alpha(t)\, q^{\alpha'}(t') \ldots\}^* = \{\ldots q^{\alpha'}(t')\, q^\alpha(t)\}. \tag{4}$$

Such variables, when endowed also with the important property (8), will be called anticlassical variables, and when dealing with them the remarks just made concerning differentiability are no longer sufficiently restrictive except, of course, where the operator D is concerned. Considerations which are special to the case of anticlassical variables are set out in § 3, which may be skipped by any reader interested only in applications to the standard variational calculus and to classical mechanics.

We shall develop the treatments of classical and anticlassical systems jointly, and where a difference of sign occurs as between the two the upper sign will refer to the classical case, in which the $q^\alpha(t)$ are ordinary arithmetical variables.

The Euler–Lagrange equations

The infinitesimal action variation $\delta\mathbf{A}_1$ is induced by infinitesimal real variations $\delta q^\alpha(t)$ which are arbitrary save for the restrictions

$$\delta \mathrm{D}^r q^\alpha(t_1) = \delta \mathrm{D}^r q^\alpha(t_2) = 0 \quad (r = 0, 1, \ldots, N-1), \tag{5}$$

and which are either arithmetical or anticlassical according to the type of system. The Euler–Lagrange equations, which express the stationarity property (1) under conditions (5) are,

$$\sum_{r=0}^{N} (-\mathrm{D})^r \frac{\partial L_1}{\partial(\mathrm{D}^r q^\alpha)} = 0 \quad (\alpha = 1, 2, \ldots, \nu). \tag{6}$$

In deriving these equations for the anticlassical case it is necessary to make explicit appeal to the postulated quotient rule (8).

The method of causal analysis

Since there is only one independent variable t it is always possible to regard the system (6) as a system of equations of motion, which specify certain of the higher time derivatives of the q, and so define or partly define a causal evolution of the q with respect to t. The highest time derivatives that can ever appear in (6) are the $\mathrm{D}^{2N}q$, and these appear linearly if at all. They will not appear if L_1 happens to depend only linearly on the $\mathrm{D}^N q$. Thus it is always permissible to regard the system (6) as a system for the determination of the $\mathrm{D}^M q^\alpha$, where M is a fixed positive integer which need never be chosen larger than $2N$, and which may with advantage be chosen smaller than this in some circumstances.

The system (6) contains nominally as many equations as there are q. In the simplest cases these equations are so constituted as to permit all the $\mathrm{D}^M q$ to be expressed directly in term of the $\mathrm{D}^{M-1}q, \ldots, \mathrm{D}^0 q$ by the ordinary processes of elimination. In such cases the $\mathrm{D}^{M-1}q, \ldots, \mathrm{D}^0 q$ can be

assigned freely and at will at some chosen and fixed instant of time. Once they have been assigned the future and past evolution of the orbit functions $q(t)$ is determined for all t by the familiar process of step-by-step integration through time, and it is plain to see that (6) is obeyed along the orbit so constructed.

The present work was originally prompted by the prevalence in modern physics of systems of a more difficult nature, for which the attempt to solve for the $\mathrm{D}^M q$ leads, by systematic elimination, to relations of the form

$$\psi(t, q^\alpha(t), \mathrm{D}q^\alpha(t), \ldots, \mathrm{D}^{M-1}q^\alpha(t)) = 0, \tag{7}$$

in which differentiable functions ψ not involving any of the $\mathrm{D}^M q$ are equated to zero. We shall call these relations the primary equations of constraint, and the corresponding functions ψ the primary constraint functions.

The chain structure of the constraint system

The primary equations of constraint apply at each instant of time, since they are derived from the Euler–Lagrange equations (6), which themselves apply at each instant. It is therefore permissible and appropriate to apply the operator D to each primary equation of constraint. This may lead to further information about the $\mathrm{D}^M q$ and also, by systematic elimination of the $\mathrm{D}^M q$ between all the members of the augmented set of equations in which they now appear, to the equation to zero of further functions ψ not involving the $D^M q$. Any new equation of constraint found in this way, and not an algebraic consequence of the primary equations of constraint, is called a secondary equation of constraint. The operator D may be applied in like manner to these, and so forth. Thus chains of primary, secondary, tertiary, etc. constraints are developed and, along with them, information about the $\mathrm{D}^M q$ is accumulated. At some stage in the development an unavoidable contradiction of the type $0 = 1$ may arise (as, for example, for the simple case $L_1 = q$). If this happens the action principle (1) has no stationary curves, and is void and empty. But if (1) is solvable then it must happen that at some stage of the constraint chain development the latest set of constraints can be inferred by algebraic manipulation (i.e. without the aid of D) from those obtained in earlier stages. The constraint chain then terminates. Termination is the only alternative to inconsistency, since the number of inequivalent constraints that can be consistently accommodated is limited by the number of variables $\mathrm{D}^{M-1}q^\alpha, \ldots, \mathrm{D}^0 q^\alpha$, which is finite.

Sufficiency of the causal chain analysis

Suppose that at some fixed instant t we assign to the variables just mentioned a set of values in accord with all the constraints. Then every constraint member of the original equations of motion (6) is satisfied at that instant. Also, since every constraint has arisen by systematic and methodical elimination of the $\mathrm{D}^M q$ between various equations involving the $\mathrm{D}^M q$, it follows that the full set of all such equations is compatible, and admits one or more solutions for the $\mathrm{D}^M q$. By taking any one such at the instant considered we satisfy all the remaining members of (6) at that instant, because the algorithm for constraint chain construction nominally includes every member of (6) among the said set of equations for the $\mathrm{D}^M q$. Using the solution $\mathrm{D}^M q$ we can proceed to construct a set of values for $\mathrm{D}^{M-1}q, \ldots, \mathrm{D}^0 q$ at time $t + \mathrm{d}t$, by the integration formula

$$\mathrm{D}^r q^\alpha(t + \mathrm{d}t) = \mathrm{D}^r q^\alpha(t) + \mathrm{D}^{r+1} q^\alpha(t)\, \mathrm{d}t \quad (r = 0, 1, \ldots, M-1).$$

Now the equations of constraint and of motion at any given stage of the constraint hierarchy express, along with all their various predecessors, the full set of conditions necessary for the

preservation through time of all the constraints of the previous stages. Thus the variables inferred for the time $t+\mathrm{d}t$ will automatically obey the full set of constraints at time $t+\mathrm{d}t$. Hence, recycling the suggested construction, we see that we can build up a solution to (6) over an extended interval of time.

Thus the constraint chain system, developed by the systematic procedure of elimination and differentiation outlined above, completely exhausts the kinematic content of (6), and the set of equations brought out by this procedure and involving the $\mathrm{D}^M q$ completely exhausts the dynamic content.

Connexion with Dirac's system of constraints

The theory of constraint chains in dynamics was first developed in a general way by Dirac (1950, 1951, 1958, 1964*a*), to whom the above causal analysis is due. However, Dirac eliminated the velocity variables $\mathrm{D}q$ in favour of certain momentum variables p. This methodological difference makes it somewhat difficult to translate directly between his formalism and that of the present paper. In particular, it is noteworthy that all of Dirac's primary constraints (his 'ϕ equations') and some of his secondary, etc., constraints (his 'χ equations') are extra to the members ψ of our constraint hierarchy.

Necessity for a holistic treatment

It is an inescapable feature of the situation that the system of equations of motion (6) cannot be causally analysed except by introducing the concept of constraint and the concept of the 'genuine' equations of motion for the $\mathrm{D}^M q$. It is however equally inescapable that there can be no absolute distinction between these two concepts. An equation which is reckoned as a 'genuine' equation of motion for one choice of M will become an equation of constraint if M be increased sufficiently. It is only the combined system of 'genuine' equations of motion and constraints all together that has any invariant significance. The methodology of the present paper succeeds only because it recognizes and exploits this close connexion between kinematics and dynamics, and does not attempt to discuss the constraints in isolation from the full dynamic problem which generates them.

The orbit manifold

If the equations found for the $\mathrm{D}^M q$ suffice to determine the $\mathrm{D}^M q$ in terms of the $\mathrm{D}^{M-1}q, \ldots, \mathrm{D}^0 q$, then the system is deterministic, and one orbit passes through each point $(t, \mathrm{D}^{M-1}q, \ldots, \mathrm{D}^0 q)$ obeying the constraints $\psi = 0$. The set of all such constrained points, here envisaged as embedded in an $(M\nu+1)$-dimensional space, constitutes a differentiable manifold, called the orbit manifold. The same concept of orbit manifold extends to systems which are only partially deterministic, or even indeterministic (e.g. $L_1 = \mathrm{D}q$), but for them a continuous infinity of orbits passes through each point of the manifold. It should be noted that in the case of anticlassical systems the embedding space has $M\nu$ anticlassical dimensions and one arithmetic dimension.

The problem of alternative choices

If the Lagrangian is a quadratic function of the q and their various time derivatives then the procedures of elimination sketched out in our discussion of the chain structure of the constraint system are quite unambiguous in execution, since all the equations to be treated are linear. However, it is our intention to give a discussion valid also for general non-linear systems.

Now the constraint chain development can always be arranged (e.g. by choosing $M \geqslant 2N$) so that all the equations involving the $\mathrm{D}^M q$ involve them linearly. Thus the repeated eliminations

which are necessary can in all cases be accomplished by the standard methods of linear algebra. However, it is necessary to bear always in mind that the various determinants encountered during the process of elimination may vanish upon certain submanifolds (something, of course, which cannot happen with a quadratic Lagrangian). The assumption at one stage of the chain development that a certain determinant is not zero may lead to a contradiction at a later stage. Should this happen the contradiction must be avoided by making the contrary assumption at the earlier stage (thereby introducing an extra constraint at that stage). The remainder of the constraint chain structure must then be explored afresh.

Obviously it is not possible to enumerate and follow through in any general way the totality of alternative options which might arise, only to be discarded later, during the constraint chain development of a general non-linear problem. It is therefore important to realize that these complications arise outside the orbit manifold, while the theorems to be proved are set within the manifold or, in the case of the constructive theorem for the Lagrange function, within the manifold and its infinitesimal neighbourhood. This observation has profound implications for the method of proof adopted for the fundamental theorem of rank and nullity. In that proof we shall avoid all reference to points not in the orbit manifold or its infinitesimal neighbourhood, by supposing that that manifold has been explicitly delineated by the methods described before we begin the proof. Any attempt to work through the whole dynamic problem directly from (6) as starting point, and without this prior knowledge, would of necessity encounter the great difficulties described, which have nothing to do with the properties we wish to establish.

Systems admitting several viable alternative choices

Lagrange functions can be constructed for which several inequivalent assumptions as to the vanishing or non-vanishing of the relevant determinants are permissible, leading to several viable but distinct differentiable orbit manifolds (e.g. $L_1 = (q^1)^2 \mathrm{D}q^2$). In such cases the various manifolds must be treated as disjoint, even though they may intersect at some places. The theorems of this paper have nothing to say with regard to any relations which might accidentally subsist between the orbits of one manifold and those of another at or near a point of intersection.

3. Anticommutative differential geometry

This section develops various concepts peculiar to the case where the dependent variables mutually anticommute, as in equation (3). It may be skipped by readers whose interest is in the classical calculus of variations.

The quotient rule for anticlassical variables

In order to resolve certain questions of uniqueness, etc., it is necessary to suppose that the various anticommutative entities B which appear in an anticlassical variational problem have the property that

$$\{FB = 0,\ \forall B\} \Rightarrow \{F = 0\}, \tag{8}$$

where F denotes any finite mathematical object.

Functions of anticlassical variables

The operational rule (8) makes it impossible to realize anticlassical variables or anticlassical constants in concrete, finite and self-contained arithmetical terms (Allcock 1973), and therefore

it is necessary to seek their significance entirely in the operations which are wrought with them. This raises the question as to what we might mean by a function of them, such as L_1. We shall take it for granted that the only interesting functions of anticlassical variables take the form of multiple power series, with complex arithmetical coefficients which might depend upon t in some way. Thus differentiability to finite order is of necessity supplanted by analytic dependence, so far as concerns any dependence on anticlassical variables. Moreover, it will be clear from (3) that any product containing the same anticlassical factor twice vanishes, and that any which does not vanish changes sign under mutual exchange of any two of its elementary factors. From this it follows that any function which, like L_1, contains only a finite number of anticlassical arguments, such as $q^1, \ldots, D^N q^\nu$ and various anticlassical constants, can be expressed uniquely as a polynomial in these arguments, with antisymmetric t-dependent complex arithmetical coefficients. Therefore the whole theory of anticlassical dynamical systems could be developed by algebraic methods in which only these arithmetical coefficients would appear. However, such an approach, although thoroughly concrete and finite, would greatly obscure the underlying structure. It is better to develop the theory of such systems in a geometric language which, using the anticlassical variables as a notational catalyst, parallels that naturally applicable to classical systems. We shall use the latter approach at all stages of this work.

Anticlassical manifolds

The geometric concept of the *shape* of a real function of anticlassical variables resides in the unique antisymmetric coefficients of the polynomial that represents it. The shape may also be explored in the presence of a real algebraic equation of constraint, such, for example, as the real equation $q^1 = 2q^2 + 3\mathrm{D}q^3 + 4\mathrm{i}q^4q^5q^6$ ($\mathrm{i}^* = -\mathrm{i}$). Two real functions of differing shape may become of equal shape modulo such a real constraint. In this way the concept of analytic manifold extends naturally to anticlassical variables, and is expressed by equalities between real polynomials of differing shapes.

In principle it is perfectly permissible to equate an even polynomial $\mathscr{E}$ to an odd polynomial $\mathscr{O}$. However, it is a simple consequence of (3) and (8) that any single equation of this sort implies the two separate equations $\mathscr{E} = 0$, $\mathscr{O} = 0$. To avoid this unnecessary source of confusion it is best therefore to adopt always the rule that even polynomials shall be equated only to even polynomials, and odd only to odd. By the same token, we are never under any circumstance allowed to equate anticlassical variables to ordinary numbers (zero excepted). Therefore the geometric concept of a definite fixed point in anticlassical space is to be defined always relative to other anticlassical parameters or anticlassical constants or odd products thereof, and the concept of *value* of a function at a fixed point in anticlassical space must be regarded as being defined in terms of these extraneous parameters through the antisymmetric arithmetical coefficients of the appropriate polynomial. In short, the concept of *value* on a zero-dimensional submanifold must be treated in quite the same way as is that of *shape* on a submanifold of higher dimension.

With these understandings it is meaningful to speak freely of points, curves, surfaces, etc., in anticlassical space, and even to extend to that space various powerful theorems of the integral calculus, such as the integrability theorems of Lie field systems, and the theorems of Stokes, Poincaré and de Rham (cf. § 11), all of which make use of such geometric concepts.

With regard to the action principle (1) it will be clear from the above discussion that the Lagrange function L_1 must, in the anticlassical case, have a definite parity with respect to the anticlassical variables and constants. For otherwise we would have two simultaneous action

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principles, and not one. Throughout the following it will be taken as axiomatic that, in the anticlassical case, L_1 is to be even. This is the only case with physical applications, and in the full ramifications of the theory (Allcock 1975) the evenness of L_1 is of decisive import.

Anticlassical differentiation

The methods of the differential calculus carry over with ease to functions of anticlassical variables, provided one adopts some suitable convention to regulate the otherwise undetermined order of factors and associated matters of sign. The ordering convention used throughout the present work is that for any function F, even or odd, we define the derivatives by the rule

$$\mathrm{d}F = \frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial F}{\partial q^\alpha}\mathrm{d}q^\alpha + \frac{\partial F}{\partial(\mathrm{D}q^\alpha)}\mathrm{d}(\mathrm{D}q^\alpha) + \ldots, \tag{9}$$

wherein all differentials d stand to the right. The derivatives so defined are unique, by (8).

Let $\mathscr{E}$ and $\mathscr{O}$ denote, as before, an even function and an odd function. Then the product rule of the calculus branches out into two rules

$$\left.\begin{aligned} \frac{\partial}{\partial q^\alpha}(F\mathscr{E}) &\equiv (F\mathscr{E})_{,\alpha} = F_{,\alpha}\mathscr{E} + F\mathscr{E}_{,\alpha}, \text{ etc.,} \\ \frac{\partial}{\partial q^\alpha}(F\mathscr{O}) &\equiv (F\mathscr{O})_{,\alpha} = -F_{,\alpha}\mathscr{O} + F\mathscr{O}_{,\alpha}, \text{ etc.} \end{aligned}\right\} \tag{10}$$

It is an immediate and notable consequence of (10) that in any anticlassical context we have

$$\frac{\partial^2}{\partial q^\alpha \partial q^\beta} = -\frac{\partial^2}{\partial q^\beta \partial q^\alpha}. \tag{11}$$

Mixed systems

In mixed systems, where classical and anticlassical variables are coupled, the classical variables perforce lose their previous arithmetical character, and become just as symbolic as the other variables. Thus the Lagrange function of a mixed system with interaction between its parts has to be fully analytic with respect to all the dependent variables $q^1, \ldots, \mathrm{D}^N q^\nu$, as well as being even and analytic in those arguments which are anticlassical. Granted this minor point, however, the theory of mixed systems is very close to that developed in the following pages for pure systems, and merits no separate discussion. The way in which the various fundamental theorems extend to mixed systems will indeed be quite self-evident.

4. The differential orbit neighbourhood of an orbit and the Jacobi equations

Orbit differentials and singular action principles

Let $\hat{q}^\alpha(t)$ denote a given orbit, and $\hat{q}^\alpha(t) + \Delta q^\alpha(t)$ another orbit infinitesimally close to it, with the instantaneous *orbit differentials* $\Delta q^\alpha(t)$ classical or anticlassical as the case may be. It is clear that in the concept Δq one implies at least that (6) must be satisfied under the substitution $q^\alpha(t) \to \hat{q}^\alpha(t)$ and also, to first order in the differentials Δq, under the substitution

$$q^\alpha(t) \to \hat{q}^\alpha(t) + \Delta q^\alpha(t).$$

However, there can be more to the situation than this, as is rendered evident even by such simple cases as $L_1 = q^{\frac{5}{2}}$ or $L_1 = q^3$, for example. For these two illustrative Lagrange functions, as for the Lagrange function $L_1 = q^2$, the only orbit is $\hat{q} = 0$, and therefore the only permissible Δq is the zero infinitesimal. But the substitution $q \to \hat{q} + \Delta q$ in the l.h.s. of the Euler–Lagrange equation (6),

with subsequent retention of the first order term only, fails to yield this or any other limitation on the infinitesimal Δq in the first two examples given.

These two examples illustrate the concept of a singular variational problem, to which we referred in § 1. The naive first order criterion for Δq fails because the second differential coefficient of L_1 degenerates on the orbit. Thus, if we wish to use a limiting process to recover our previous conclusion that the only possible orbit differential Δq is zero for these two singular Lagrangian systems, then we must use some limiting process involving explicitly certain properties of terms of order of smallness higher than the first. It should hardly be necessary to point out that the special difficulty which has arisen here over the definition of the orbit differential Δq^α does not arise in the ordinary problems of the calculus, where one is concerned merely to define the differential $df(q^1, q^2, \ldots)$ of a differentiable function f. That much simpler infinitesimal concept can always be given complete and rigorous meaning as a limiting process whose end result refers only to terms of the first order.

We shall now introduce a limiting process of higher order, sufficiently delicate to encompass properly cases such as those just described.

We make the replacement
$$\Delta q^\alpha(t) \to \epsilon\eta^\alpha(t) + \vartheta^\alpha(\epsilon, t), \tag{12}$$
where the $\eta^\alpha(t)$ are independent of ϵ and ϵ is to tend to zero, and where the $\vartheta^\alpha(\epsilon, t)$ are discrepancy variables which are to tend to zero faster than ϵ. We may then say that the $\eta^\alpha(t)$ characterize a one dimensional *differential orbit neighbourhood* of the orbit $\hat{q}^\alpha(t)$ if, for every ϵ smaller in magnitude than some non-zero ϵ_0, we can find a set of $\vartheta^\alpha(\epsilon, t)$ such that both

$$\left.\begin{aligned} &\text{(i)} \quad \sum_{r=0}^{N} (-\mathrm{D})^r \frac{\partial L_1}{\partial(\mathrm{D}^r q^\alpha)}\bigg|_{q\to\hat{q}+\epsilon\eta+\vartheta} = 0, \text{ exactly,} \\ &\text{and (ii)} \quad \epsilon^{-1}\vartheta(\epsilon, t) \text{ tends uniformly to zero as } \epsilon \text{ tends to zero.} \end{aligned}\right\} \tag{13}$$

And, when (13) are both obeyed, we are entitled to make the substitution
$$\Delta q^\alpha(t) = \epsilon\eta^\alpha(t) \tag{14}$$
in any expression wherein a limit $\epsilon \to 0$ is implied, provided that in that expression it is required to work only to first order in ϵ.

Equations (13) and (14) and the stated operational content of (14) define the orbit differential $\Delta q^\alpha(t)$ for the one dimensional differential orbit neighbourhood associated with η^α, and they define it directly in terms of the equations of motion (6) both for regular and singular cases. Now the causal constraint chain analysis of § 2 indicates the possibility of an alternative but equivalent definition, based upon the fact that the orbit manifolds of the action principle are characterized by the vanishing of a set of differentiable constraint functions ψ. From this it follows without further analysis that the solutions η^α of the system (13) can be compounded linearly, except at points of intersection of distinct orbit manifolds. As already intimated, we shall not prove any properties of such points of intersection.

The Jacobi equations

Subject to certain modest requirements of differentiability (which would *not* be met, for example, by such a Lagrange function as $L_1 = q^{\frac{3}{2}}$, which is singular in a manner over and above the type of singularity so far discussed), the solutions η of the proper defining equations (13) always obey the following limiting system linear in η, and called the Jacobi equations
$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\left\{\sum_{r=0}^{N} (-\mathrm{D})^r \frac{\partial L_1}{\partial(\mathrm{D}^r q^\alpha)}\bigg|_{q\to\hat{q}+\epsilon\eta}\right\} = 0. \tag{15}$$

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However, as we have just seen, it is possible to have Lagrange functions which are sufficiently differentiable to ensure that the limit in (15) exists for any η, but which are nevertheless singular in the sense that not all solutions η of (15) obey the proper defining equations (13) of an orbit differential.

It will be proved in § 11 that if an orbit system admits a Lagrange bracket structure having the normal properties of rank and nullity, then its complete and most advantageous description involves a regular Lagrange function. Singular Lagrange functions, having the defective feature that (15) $\not\Rightarrow$ (13), give also a defective formulary for the Lagrange bracket structure, and can always be avoided when there exists a Lagrange bracket with the properties indicated. Orbit systems with unavoidably subnormal rank properties are also of great interest, however (as, for example, in relativistic electrodynamics). We shall indicate that for such cases one can usually embed the orbits into those of a larger regular Lagrangian system by relaxing some of the constraints, or alternatively achieve regularity by relaxing some of the equations of motion while preserving such deterministic aspects as are physically relevant.

In view of these possibilities it seems plausible to dismiss singular action principles as being of no interest, and as being incorrectly designed for the orbits to which they apply. If this judgement is correct, then it justifies fully the method of the present paper, which at heart comprises a detailed analysis of the Jacobi equations (15). Alternatively, one might argue that it is always a good idea to start by studying (15), because even for singular systems, where they are not completely definitive, they are still in no way incorrect.

The action principle for the Jacobi equations

The Jacobi equations (15) assert that the locus $q(t) = \hat{q}(t) + \epsilon\eta(t)$ renders the l.h.s. of (6) zero to within a term which vanishes with ϵ faster than does ϵ itself. Therefore if we evaluate the action $\mathbf{A}$ for this locus q, and induce an infinitesimal variation of q by varying η finitely, so that $\delta q = \epsilon\,\delta\eta$, then the variation $\delta\mathbf{A}$ will vanish with ϵ faster than does ϵ^2. Moreover $\hat{q}$ obeys (6) exactly, and therefore that part of $\mathbf{A}$ which is linear in $\epsilon\eta$ is the integral of a perfect differential, and makes no contribution to $\delta\mathbf{A}$. Hence (15) may be obtained by varying η in the quadratic action principle

$$\delta\mathbf{J}_1 \equiv \delta\int_{t_1}^{t_2}\frac{1}{\epsilon^2}[\text{quadratic part of } \{L_1(t, q, \ldots, \mathrm{D}^N q)|_{q\to\hat{q}+\epsilon\eta}\}]\,\mathrm{d}t = 0. \tag{16}$$

From this it will be seen that the Jacobi equations are associated entirely with the *second variation* of the action integral $\mathbf{A}$.

5. Standard forms for the action principle and its Jacobi equations

Elimination of higher derivatives

The proof of coordinate-invariant theorems may often be greatly expedited by the use of special coordinate systems at intermediate stages. This is certainly so here. A considerable economy of technique may be effected by introducing extra variables to represent the time derivatives which appear in L_1. We define collectively a new and extended set of variables

$$q^{ar} \equiv \mathrm{D}^r q^a \quad (r = 0, 1, \ldots, 2N-1), \tag{17}$$

and treat all of them henceforth as coordinates on the same footing. The action principle (1) then appears in the guise of a constrained variational problem, in which both the coordinates $q^{\alpha r}$ and their variations are to respect the set of variational constraints (17).

In a familiar way we substitute for this an entirely equivalent variational problem in which all conceivable variations are free inside the interval $[t_1, t_2]$, and in which the variational constraints (17) are incorporated with the help of a set of Lagrange multipliers $\pi_{\alpha r}$ $(r = 0, 1, \ldots, 2N-2)$, to be put on the same footing as the $q^{\alpha r}$ and varied with them. The action principle (1) then assumes the equivalent quasi-linear first order form

$$\delta \int_{t_1}^{t_2} L_2(t, q^{\alpha 0}, \ldots, q^{\alpha\, 2N-1}, \pi_{\alpha 0}, \ldots, \pi_{\alpha\, 2N-2}, \mathrm{D}q^{\alpha 0}, \ldots, \mathrm{D}q^{\alpha\, 2N-2})\, \mathrm{d}t = 0, \tag{18}$$

where

$$L_2 = \sum_{r=0}^{2N-2} \pi_{\alpha r}(\mathrm{D}q^{\alpha r} - q^{\alpha\, r+1}) + L_1(t, q^{\alpha 0}, q^{\alpha 1}, \ldots, q^{\alpha N}). \tag{19}$$

In the case of anticlassical systems the Lagrange multipliers $\pi_{\alpha r}$ are to be anticlassical imaginary, to preserve the reality and evenness of L_2 and its variations.

With the above simple procedure any action principle whatever can be cast into a form which involves no time derivatives save the first, and these only linearly. Admittedly, if L_1 is already quasi-linear and of first order then the procedure clutters the problem with a lot of unnecessary Lagrange multipliers. And again, if L_1 is one of the Lagrangians of ordinary classical particle mechanics then Hamilton's method of linearization improves on ours by involving only two-thirds as many variables. So the procedure suggested is not an economic one. We shall find a better one, which furnishes a non-canonical generalization of that of Hamilton, in §10 (see especially equation (134)). But at the moment we aim merely for simplicity and universality, and since in these respects our procedure could hardly be bettered, we adopt it henceforth as standard.

The system of equations of motion obtained from (18) and (19) can be broken into two sets, one for the q and one for the π, as follows

$$\left.\begin{aligned} \mathrm{D}q^{\alpha 0} &= q^{\alpha 1}, \\ &\cdots\cdots\cdots\cdots \\ \mathrm{D}q^{\alpha\, 2N-2} &= q^{\alpha\, 2N-1}, \end{aligned}\right\} \tag{20a}$$

$$\left.\begin{aligned} \mathrm{D}\pi_{\alpha 0} &= \partial L_1/\partial q^{\alpha 0}, \\ \mathrm{D}\pi_{\alpha 1} &= \partial L_1/\partial q^{\alpha 1} - \pi_{\alpha 0}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathrm{D}\pi_{\alpha\, 2N-2} &= \partial L_1/\partial q^{\alpha\, 2N-2} - \pi_{\alpha\, 2N-3}, \\ 0 &= \partial L_1/\partial q^{\alpha\, 2N-1} - \pi_{\alpha\, 2N-2}. \end{aligned}\right\} \tag{20b}$$

The variational constraints (17) of the constrained action principle mentioned in the sentence following (17) now appear disguised as equations of motion (20 *a*). A set of new primary constraints is also in evidence, in the last of equations (20 *b*). But the constraints of the original action principle (1) cannot be seen directly in (20), and arise only in the higher stages of the constraint chains generated by the last of (20 *b*).

The generalized momenta of an orbit

We can recover equations (6) from equations (20) by applying $\mathrm{D}^0, -\mathrm{D}^1, \mathrm{D}^2, \ldots, -\mathrm{D}^{2N-1}$ respectively to the members of (20 *b*), and adding to eliminate the π, using (20 *a*) where necessary.

Alternatively, by reading (20 *b*) upwards one by one and using (20 *a*) where necessary, we find that the Lagrange multipliers π along any orbit are given by

$$\pi_{\alpha r} = \sum_{s=r+1}^{2N-1} (-\mathrm{D})^{s-r-1} \frac{\partial L_1(t, q, \ldots, \mathrm{D}^N q)}{\partial(\mathrm{D}^s q^\alpha)} \equiv p_{\alpha r}(t, q^{\beta 0}, \ldots, q^{\beta\, 2N-1}) \quad (r = 0, 1, \ldots, 2N-2). \tag{21}$$

Provided we stay in the orbit manifold the $\pi_{\alpha r}$ are completely fixed by (21), and therefore on orbit they can be regarded as auxiliary variables dependent upon and determined by the $\mathrm{D}^r q$.

The functions $p_{\alpha r}$ introduced and defined by (21) have a vital part to play in the construction of the Lagrange bracket. We call them the generalized momenta of the Lagrange function L_1. It is noteworthy that any $p_{\alpha r}$ with $r > N-1$ is identically zero, and that the $p_{\alpha r}$ depend, at most, on the variables explicitly indicated in (21).

The general quasi-linear first order action principle

We now introduce a still larger set of real coordinates q^a to stand collectively for all the real variables $q^{\alpha r}$ and $\pi_{\alpha r}$ ($\mathrm{i}\pi_{\alpha r}$ in the anticlassical case) which are varied in (18). Then (18) can be subsumed under the most general possible class of quasi-linear first-order action principles

$$\delta \int_{t_1}^{t_2} L_3 \,\mathrm{d}t \equiv \delta \int_{t_1}^{t_2} \{p_a(t, q)\, \mathrm{D}q^a + p_t(t, q)\} \,\mathrm{d}t = 0, \tag{22}$$

from which it can be recovered by suitably specializing the functions p_a and p_t. It is convenient to base further analysis on this generalized class, using the compact notation which it presents.

The connexion with Pfaff's problem

There is a subtle distinction between (22) and the Pfaffian variational problem

$$\delta \int \{p_a(t, q)\, \mathrm{d}q^a + p_t(t, q)\, \mathrm{d}t\} = 0, \tag{23}$$

which makes it impossible to use the standard literature on Pfaff's problem (Forsyth 1890; Goursat 1922*a*; Thomas 1937; Cartan 1946; Schouten & Kulk 1949*a*; Slebodzinski 1963; Carathéodory 1965) to resolve in any direct way the problem posed by (22). For in (22) the very notation itself already presupposes that we deal with extremals such that $\mathrm{d}t \neq 0$ for every segment, while (23) embodies no such proviso. We shall study this curious but significant distinction more closely.

Let us define

$$\left.\begin{aligned} A_{ab} &\equiv \partial p_a/\partial q^b \mp \partial p_b/\partial q^a = \mp A_{ba}, \\ A_{at} &\equiv \partial p_a/\partial t - \partial p_t/\partial q^a \equiv -A_{ta}, \end{aligned}\right\} \tag{24}$$

where upper and lower signs refer respectively to ordinary and to anticlassical systems. Then the extremals of (22) obey

$$A_{ab}\, \mathrm{D}q^b + A_{at} = 0. \tag{25}$$

For anticlassical systems the p_a are odd imaginary and p_t is even real. Therefore, in view of the stated properties of antisymmetry–symmetry in (24), the premultiplication of (25) by $\mathrm{D}q^a$ yields for both types of system the 'energy equation'

$$A_{tb}\, \mathrm{D}q^b = 0. \tag{26}$$

In contrast to (25) and (26) the extremals of the Pfaffian variational problem (23) are loci along which

$$\left.\begin{aligned} \mathsf{A}_{ab}\,\mathrm{d}q^b + \mathsf{A}_{at}\,\mathrm{d}t = 0, \\ \mathsf{A}_{tb}\,\mathrm{d}q^b = 0. \end{aligned}\right\} \tag{27}$$

Now if the submatrix A_{ab} is invertible then no non-trivial solution of (27) can have $\mathrm{d}t$ equal to zero, and we can divide by $\mathrm{d}t$ to recover (25) and (26). But if A_{ab} has no inverse then (27) may have solutions with $\mathrm{d}t = 0$, over and above any solutions belonging to (25) and (26).

As a simple example of this state of affairs, let us take a particle 'anchored' at the origin of two dimensional space, with $L_3 = (q^1)^2 + (q^2)^2$. Then equations (25) yield $q^1 = q^2 = 0$. But (27) yield only $q^1\,\mathrm{d}t = q^2\,\mathrm{d}t = q^1\,\mathrm{d}q^1 + q^2\,\mathrm{d}q^2 = 0$, and have therefore non-trivial solutions with $\mathrm{d}t = 0$, $q^1 \neq 0$, $q^2 \neq 0$, which have nothing to do with the dynamical problem. The usual theory of Pfaff variational problems would in this example apply directly to these unwanted solutions, for which the rank of the full three dimensional matrix A is equal to two. But the solutions of interest to us are degenerate from this point of view, since for them the rank of the full matrix A is equal to zero.

Degeneration of the full matrix A of the sort just illustrated is the rule rather than the exception for systems (22) having constraint chains. The solutions of (22) can therefore only occasionally be encompassed within the scope of standard Pfaff theory since that theory, with its procedures for the canonical reduction of the linear differential form in (23), always fails at points of degeneration of A. It should be noted that the type of degeneration to which we now refer is distinct from and additional to the various sorts of degeneration encountered in our discussion of the problem of alternative choices (§2), and also from the other sort of degeneration which is met when the Lagrange function is singular (§4). The latter are only incidental to our main problem.

The standard form for the Jacobi equations

In the spirit of equation (14) we now write

$$\Delta q^a(t) = \epsilon\eta^a(t), \tag{28}$$

where, just for the immediate moment, we understand the symbols Δq^a to refer to arbitrary differential displacements, and not merely to differential displacements from orbit to orbit. With the above notation the quadratic action principle (16) for the Jacobi equations (15) of the general quasi-linear first-order system (22) reads as follows

$$\delta\mathbf{J}_3 \equiv \delta\int_{t_1}^{t_2} \tfrac{1}{2}\{\mathsf{A}_{ab}\,\eta^b\,\mathrm{D}\eta^a + \mathsf{B}_{ab}\,\eta^b\eta^a\}\,\mathrm{d}t = 0, \tag{29}$$

where A_{ab} is defined by (24), and

$$\mathsf{B}_{ab} \equiv \frac{\partial}{\partial q^b}\left(\frac{\partial p_c}{\partial q^a}\right)\mathrm{D}q^c + \frac{\partial}{\partial q^b}\left(\frac{\partial p_t}{\partial q^a}\right) - \tfrac{1}{2}\mathrm{D}\left(\frac{\partial p_a}{\partial q^b} \pm \frac{\partial p_b}{\partial q^a}\right) = \pm\,\mathsf{B}_{ba}, \tag{30}$$

and where, in the derivation from (16), a perfect differential $-\tfrac{1}{2}\mathrm{D}\{(\partial p_a/\partial q^b)\,\eta^b\eta^a\}$ has been added to the integrand to simplify its overall structure.

From (29) it is at once apparent that the Jacobi equations of (22) are

$$\mathsf{A}_{ab}\,\mathrm{D}\eta^b = (\mathsf{B}_{ab} - \tfrac{1}{2}\mathrm{D}\,\mathsf{A}_{ab})\,\eta^b. \tag{31}$$

In these equations the matrices A and B are evaluated on a fixed base orbit $\hat{q}$, and if we confine our attention wholly to one such fixed base orbit they can be regarded as functions of t only.

6. Lagrange brackets

The conserved antisymmetric bilinear differential form

Let η_1 and η_2 be two solutions of (31) on the same base orbit $\hat{q}$, and let $\Delta_1 q$ and $\Delta_2 q$ be the associated differential displacements of orbit to orbit, so that

$$\Delta_1 q^a(t) = \epsilon \eta_1{}^a(t), \quad \Delta_2 q^a(t) = \epsilon \eta_2{}^a(t). \tag{32}$$

Then from Δ_1 and Δ_2 we can construct a certain bilinear differential form, antisymmetric under exchange of Δ_1 with Δ_2

$$\Delta_1 \times \Delta_2 \equiv A_{ab} \Delta_2 q^b \Delta_1 q^a = -\Delta_2 \times \Delta_1. \tag{33}$$

This differential form enjoys the conservation property

$$\mathrm{D}(\Delta_1 \times \Delta_2) = 0, \tag{34}$$

as may readily be verified by using the Jacobi equations (31) and the symmetry–antisymmetry properties (24) and (30).

Apart from the trivial matter of the implicit factor ϵ^2, $\Delta_1 \times \Delta_2$ is just the Lagrange bracket of analytical mechanics. Indeed, if our standardized dynamic system has arisen from the general action principle (1) via the change of notation (17), so that its standard action principle (22) has the rather special form given in (18) and (19), then we can invert the transformation and express $\Delta_1 \times \Delta_2$ of equation (33) in terms of the variables of the original action principle. Doing this, and making carefully all the necessary identifications, we find that the conserved antisymmetric bilinear form of any action principle (1) is given by

$$\Delta_1 \times \Delta_2 \equiv \sum_{r=0}^{N-1} \{\Delta_2 p_{\alpha r} \Delta_1 q^{\alpha r} - \Delta_1 p_{\alpha r} \Delta_2 q^{\alpha r}\}, \tag{35}$$

where the $p_{\alpha r}$ are the momentum functions (21) of the same action principle, and $q^{\alpha r} \equiv \mathrm{D}^r q^\alpha$.

When the bilinear form $\Delta_1 \times \Delta_2$ is expressed in the terms just given the connexion between it and the Lagrange bracket of analytical mechanics (Poincaré 1899; Cartan 1922; Whittaker 1937; Lanczos 1949; Pars 1965; Abraham 1967) becomes completely evident. The one is precisely ϵ^2 times the other.

The osculatory properties of the Jacobi action principle

It would not be very pertinent to the main theme of this paper to demonstrate the scalar property of $\Delta_1 \times \Delta_2$ and the coordinate invariance of its general defining formulae (35) and (21). Such matters are treated elsewhere (Allcock 1975). It is however appropriate to mention that (35) and (21) constitute a general formulary invariant against changes of description involving extra variables and associated Lagrange multipliers, as in (17) and (19). In particular, it is a property of this formulary that its direct application to the Lagrange function L_3 of (22) yields precisely the differential form (33) again, as may immediately be verified by direct calculation.

With this property of invariance in mind, we can make immediately three comments

(i) the orbit manifold of the general action principle (1) maps *via* a linear and invertible substitution (17) into the orbit manifold of an equivalent quasi-linear first-order action principle (22) or (18),

(ii) the Jacobi equations (15) of (1) transform into (31) under this mapping, and (31) transform into (15) under the inverse mapping,

(iii) the bilinear differential form (defined in each case by appropriate application of formulae (35) and (21) to whichever Lagrange function is being considered) is scalar under the mapping.

Let us move next to the Jacobi action principle (29), whose Euler–Lagrange equations (31) are the Jacobi equations of (22), and in terms of which the base orbit $\hat{q}$ is represented by the solution $\hat{\eta} = 0$. It is evident that besides being the Jacobi equations of (22) and the Euler–Lagrange equations of (29), the equations (31) are also the Jacobi equations of (29) on the base orbit $\hat{\eta} = 0$. Jacobi and Euler–Lagrange equations merge their separate identities for any Lagrange function quadratic in the dependent variables, and any such Lagrange function is regular. Furthermore, the Lagrange bracket on the base orbit $\hat{\eta} = 0$ of (29) (obtained by applying to (29) the universal formulae (35) and (21), and dividing by ϵ^2 in the usual way) is nothing other than the structure

$$A_{ab}\eta_2{}^b\eta_1{}^a \tag{36}$$

already encountered in (33). Thus we have finally a fourth comment

(iv) the orbit differentials of the Jacobi equations of (1) on the base orbit $\hat{q}$ of (1) map bijectively into the orbit differentials of the Jacobi equations of (29) on the base orbit $\hat{\eta} = 0$ of (29), and the Lagrange bracket of (1) on the base orbit $\hat{q}$ of (1) is equal to the Lagrange bracket of (29) on the base orbit $\hat{\eta} = 0$ of (29).

This result completely linearizes our problem, and provides a good point of departure for the proof of the fundamental theorems of rank.

7. Proof of the theorem of rank for deterministic systems

The unifying power of the Jacobi action principle

In proving the theorem of rank it is essential to take into account that the constraints which delimit the variables η in (36) are in no way imposed arbitrarily from outside, but arise from the Jacobi equations (31). The constraint chains of (31) must be explicitly and fully explored. Only in this way is it possible to discover those influences which, flowing from the two matrices A and B, and mingling together in (31), affect jointly both the properties of determinism of (31) and the properties of rank of the bilinear form (36).

Canonical reduction of A

In the anticlassical case the matrix A of the bilinear form (36) is symmetric imaginary, and it can therefore be rendered diagonal imaginary by applying Lagrange's method of completion of squares. The Lagrange algorithm provides us with an invertible real linear transformation

$$\eta^a \to \xi^a; \quad \eta^a = \xi^b \mathsf{S}_b{}^a, \tag{37}$$

whose elements $\mathsf{S}_b{}^a$ are rational functions of those of A (and commute with those of A since the latter are even in the $\hat{q}$).

This transformation induces in A a *congruent transformation*

$$\mathsf{A}_{ab} \to \mathsf{C}_{ab} = \mathsf{S}_a{}^c \mathsf{S}_b{}^d \mathsf{A}_{cd}, \tag{38}$$

where C is diagonal and imaginary and even in the $\hat{q}$. By a further operation of rescaling every non-zero diagonal element can be reduced to $\pm$i.

For classical systems A is antisymmetric real, but again a suitable rational algorithm exists

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(Turnbull & Aitken 1945), by which, with suitable rescaling, a canonical reduction can be achieved through an invertible real linear transformation (37).

To express the canonical reduction of A we introduce a suitable partition of the range of the index a, and two indices r and s to run respectively over the two portions of the full range. For anticlassical systems the range of r may be even or odd depending on the system – in either case we obtain

$$\left.\begin{aligned} C_{rs} = C_{sr} = C_{ss'} = 0, \\ \mathrm{i}C_{rr'} = \delta_{rr'} \text{ for some } r, \quad -\delta_{rr'} \text{ for the rest.} \end{aligned}\right\} \tag{39}$$

For classical systems the range of r is always even, and we have

$$\left.\begin{aligned} C_{rs} = C_{sr} = C_{ss'} = 0, \\ C_{rr'} = \delta_{r-1\,r'} \text{ for even } r, \quad -\delta_{r+1\,r'} \text{ for odd } r. \end{aligned}\right\} \tag{40}$$

The Jacobi action principle after canonical reduction of its kinetic term

Since the elements of S are rational functions of those of A they are differentiable with respect to time over extended regions of time. The same applies to the irrational scaling factors. Therefore, referring now to the quadratic action functional $\mathbf{J}$ of (29), and expressing it in terms of the new coordinates ξ, we can complete the reduction by writing

$$B_{ab} \to E_{ab} = S_a{}^c\, S_b{}^d\, B_{cd} + \tfrac{1}{2}(\mathrm{D}S_a{}^c)\, S_b{}^d\, A_{cd} - \tfrac{1}{2} S_a{}^c (\mathrm{D}S_b{}^d)\, A_{cd} = \pm E_{ba}, \tag{41}$$

in which equation the scaling factors have been absorbed into S. The Jacobi action principle (29) now assumes the simpler form

$$\delta \mathbf{J}_3 \equiv \delta \int \tfrac{1}{2}\{ C_{rr'} \xi^{r'} \mathrm{D}\xi^r + E_{rr'} \xi^{r'} \xi^r + 2E_{rs} \xi^s \xi^r + E_{ss'} \xi^{s'} \xi^s \}\, \mathrm{d}t = 0. \tag{42}$$

The Jacobi equations (31) simplify correspondingly, and those of them which involve the $\mathrm{D}\xi^r$ can be solved for the $\mathrm{D}\xi^r$ by noting that the inverse of the regular sub-matrix $C_{rr'}$ is just $-C_{rr'}$. Using this property, we write the new Jacobi equations as

$$\mathrm{D}\xi^r = -C_{rr'}(E_{r'r''}\xi^{r''} + E_{r's}\xi^s), \tag{43}$$

$$0 = E_{sr}\xi^r + E_{ss'}\xi^{s'}. \tag{44}$$

The structure of a regular chain and the concept of pivotal element

Equations (44) are the primary constraints of the system (42). Any member of (44) for which some element $E_{ss'}$ is non-zero can be used to fix the variable $\xi^{s'}$ in terms of the other ξ^s and the ξ^r and hence, by a procedure which has yet to be described, completely eliminate that variable from the description of the system.

Suppose however that over an extended interval of time about the instant under investigation, we find that

$$E_{ss'} = 0 \quad (\forall s, s'). \tag{45}$$

In this case, equations (44) constrain the ξ^r and therefore there are associated secondary constraints, which are obtained by applying to (44) the operator D, and then using (43) where possible, as described in a general way in § 2. The secondary constraints read

$$(\mathrm{D}E_{sr})\,\xi^r - E_{sr}\, C_{rr'}(E_{r'r''}\xi^{r''} + E_{r's'}\xi^{s'}) = 0. \tag{46}$$

If some element $E_{sr}\,C_{rr'}\,E_{r's'}$ is non-zero then the relevant member of (46) can be used to fix the variable $\xi^{s'}$ in terms of the other ξ^{s} and the ξ^{r} and hence, by a procedure which has yet to be described, completely eliminate that variable from the description of the system.

However, it may happen that we find over an extended interval of time that

$$E_{sr}\,C_{rr'}\,E_{r's'} = 0 \quad (\forall s, s'). \tag{47}$$

In that case we must proceed to the tertiary constraints, and so forth.

We shall now examine a general situation, in which all the primary, secondary, ..., n-ary ($n \geqslant 1$) constraints involve only the ξ^{r}, while at least one of the $(n+1)$-ary constraints involves one or more of the ξ^{s}. We write the primary ... $(n+1)$-ary constraints respectively as

$$v_{1sr}\,\xi^{r} = 0, \ldots, v_{nsr}\,\xi^{r} = 0,\ v_{n+1\,sr}\,\xi^{r} + v_{n+1\,ss'}\,\xi^{s'} = 0. \tag{48}$$

The identification

$$v_{1sr} = E_{sr} \tag{49}$$

follows immediately from (44) and (45). The $(j+1)$-ary constraints are the time derivatives of the j-ary constraints and are given by

$$\mathrm{D}(v_{jsr}\,\xi^{r}) \equiv (\mathrm{D}v_{jsr})\,\xi^{r} - v_{jsr}\,C_{rr'}(E_{r'r''}\,\xi^{r''} + E_{r's'}\,\xi^{s'}) = 0 \quad (j = 1, \ldots, n). \tag{50}$$

It is our hypothesis that these do not depend on the ξ^{s} for $j = 1, \ldots, n-1$. Using the symmetry–antisymmetry property of equation (41) to write $E_{r's'} = \pm E_{s'r'} = \pm v_{1s'r'}$ we can express this lack of dependence on the ξ^{s} by the equations

$$v_{jsr}\,C_{rr'}\,v_{1s'r'} = 0 \quad (\forall s, s'; j = 1, \ldots, n-1). \tag{51}$$

Feeding this back into (50) we obtain the recursive formula

$$v_{j+1\,sr} = \mathrm{D}v_{jsr} - v_{jsr'}\,C_{r'r''}\,E_{r''r} \quad (j = 1, \ldots, n). \tag{52}$$

It is our hypothesis that for some particular s and some particular s', say $s = \bar{s}$ and $s' = \bar{s}'$, there exists a non-zero *pivotal element*

$$P_{n\bar{s},\,1\bar{s}'} \equiv v_{n\bar{s}r}\,C_{rr'}\,v_{1\bar{s}'r'} \equiv \mp v_{n+1\,\bar{s}\bar{s}'} \neq 0. \tag{53}$$

Because of the recursive property (52), this implies that none of the vectors $v_{1\bar{s}r}, v_{2\bar{s}r}, \ldots, v_{n\bar{s}r}$ are zero. Thus we have a chain of n constraints on the ξ^{r} and at its end one $(n+1)$-ary constraint which fixes $\xi^{\bar{s}'}$. This $(n+1)$-membered chain starts from the primary constraint $E_{\bar{s}r}\,\xi^{r} = 0$, which comes by variation of $\xi^{\bar{s}}$ in the Jacobi action principle. We must now establish certain relations which subsist between this chain and a second chain coming by variation of $\xi^{s'}$. The two chains go together.

The concept of the doubled regular chain

It is in the nature of the case that the nullity properties (45) and (51) really do hold good over an extended interval of time about the instant t under investigation, as indeed we have already assumed. For if they were to hold by accident at some particular instant of time, yet not at neighbouring instants, then that particular instant would be a singular point, through which the orbit could not be continued. Clearly, we would not wish to include singular points in our analysis. We are therefore entitled to apply to (51) the operator D. This yields the further set of nullity conditions

$$(\mathrm{D}v_{jsr})\,C_{rr'}\,v_{1s'r'} + v_{jsr}\,C_{rr'}\,\mathrm{D}v_{1s'r'} = 0 \quad (\forall s, s'; j = 1, \ldots, n-1). \tag{54}$$

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If we eliminate the Dv here by using (52) both for the chain s and for the chain s', and make appeal to properties (41) and (24), which tell us that $E^{\mathrm{T}} = \pm E$, $C^{\mathrm{T}} = \mp C$, and if we use the short notation defined in (53), we find that the identity (54) simplifies to

$$P_{js,\,2s'} = -P_{j+1\,s,\,1s'} \quad (\forall s, s'; j = 1, \ldots, n-1). \tag{55}$$

Hence, by (51)

$$P_{js,\,2s'} = 0 \quad (\forall s, s'; j = 1, \ldots, n-2). \tag{56}$$

We now apply D to each of (56) and proceed as before, and so forth. We finish with the system of identities

$$P_{js,\,ks'} = 0 \quad (\forall s, s'; j+k \leqslant n), \tag{57}$$

$$P_{js,\,n-j+1\,s'} = (-)^{n-j} P_{ns,\,1s'} \quad (\forall s, s'; j = 1, \ldots, n). \tag{58}$$

If n is odd we may set $j = \frac{1}{2}(n+1)$ in the last equation. This yields an identity involving the middle members of the chains s and s'

$$v_{\frac{1}{2}(n+1)\,sr}\, C_{rr'}\, v_{\frac{1}{2}(n+1)\,s'r'} = (-)^{\frac{1}{2}(n-1)} P_{ns,\,1s'}. \tag{59}$$

Similarly

$$v_{\frac{1}{2}(n+1)\,s'r}\, C_{rr'}\, v_{\frac{1}{2}(n+1)\,sr'} = (-)^{\frac{1}{2}(n-1)} P_{ns',\,1s}. \tag{60}$$

Since $C_{rr'} = \mp C_{r'r}$, and since also the elements of the v are even in the $\hat{q}$ in the anticlassical case, a comparison of the terms involving the middle members in (59) and (60) yields the following fundamental property of the end members of the chains

$$P_{ns,\,1s'} = \mp P_{ns',\,1s} \quad \text{if } n \text{ is odd.} \tag{61}$$

If n is even we compare the cases $j = \frac{1}{2}n$ and $j = \frac{1}{2}n+1$, interchanging s and s' in the second of them, and this yields

$$P_{ns,\,1s'} = \pm P_{ns',\,1s} \quad \text{if } n \text{ is even.} \tag{62}$$

We now have two related regular chains terminating in a pair of related and non-zero pivotal elements $P_{n\bar{s},\,1\bar{s}'}$ and $P_{n\bar{s}',\,1\bar{s}}$. Those members of this doubled chain which involve only the ξ^r are

$$v_{j\bar{s}r}\xi^r = 0, \quad v_{j\bar{s}'r}\xi^r = 0 \quad (j = 1, \ldots, n). \tag{63}$$

We do not exclude the possibility that $\bar{s}$ and $\bar{s}'$ coincide, in which case the two chains merge. By (62), (61) and (53) this can happen for ordinary systems only when n is even, and for anticlassical systems only when n is odd. If it does happen, we may say we have a *diagonal regular chain* of order $n+1$. But if all diagonal chains are singular to this order (i.e. if $P_{ns,\,1s} = 0$; $\forall s$) then the possibility remains that there is some *doubled regular chain* of order $n+1$. We proceed now to establish properties of linear independence for the two cases.

Linear independence of the members of diagonal and doubled regular chains

For a diagonal regular chain of order $n+1$ we have to ask, 'Is there any relation of linear dependence of the form

$$\sum_{j=1}^{n} x_j v_{j\bar{s}r} = 0?'$$

By multiplying this supposed relation by $C_{rr'}v_{1\bar{s}r'}$ and using (57), (58) and (53) we find that $x_n = 0$, multiplying next by $C_{rr'}v_{2\bar{s}r'}$ we find $x_{n-1} = 0$, and so forth, finishing with $x_1 = 0$.

For a doubled regular chain of order $n+1$ we have to ask, 'Is there any relation of linear dependence of the form

$$\sum_{j=1}^{n} x_j v_{j\bar{s}r} + \sum_{j=1}^{n} y_j v_{j\bar{s}'r} = 0?'$$

Provided that

$$P_{n\bar{s},\,1\bar{s}} = P_{n\bar{s}',\,1\bar{s}'} = 0, \tag{64}$$

i.e. provided that the two associated diagonal chains are singular to order $n+1$, the answer is again that there is no relation of linear dependence.

Thus a diagonal regular chain of order $n+1$ serves to eliminate simultaneously $\xi^{\bar{s}}$ and n independent ξ^r, while a doubled regular chain of order $n+1$ whose associated diagonal chains are singular to that order serves to eliminate simultaneously ξ^s, $\xi^{\bar{s}'}$ and $2n$ independent ξ^r.

The sequence of eliminations

In using the constraints to eliminate variables from the Jacobi action principle it is essential to eliminate a whole diagonal chain or a whole doubled chain in one operation, and to perform the chain eliminations in the correct sequence. Thus we give regular nth order chains precedence over regular $(n+1)$th order chains, and we give diagonal regular nth order chains precedence over doubled regular nth order chains. Every elimination alters the structure of the Jacobi action, and of the chains not yet eliminated. Thus the question of precedence must be examined afresh at each stage.

Reduction of the Jacobi action principle by prior eliminations based on a doubled regular chain

In view of the sequential principle just enunciated, it behoves us to examine a system for which the first non-zero pivots are non-diagonal, and arise in the $(n+1)$-ary stage, as in (53). The doubled chain associated with the pair of non-zero pivots can be used at each instant of time over an extended period of time to eliminate $\xi^{\bar{s}}$, $\xi^{\bar{s}'}$, and $2n$ of the ξ^r from the description of the orbits of the Jacobi action principle (42). This suggests that we try the effect of making these eliminations before varying the action. The equations of motion which would result from such a reduced variational problem may be found by the method of Lagrange multipliers. Thus our proposal is equivalent to a free variation of the ξ and the λ and the λ' in the principle

$$\delta\Big\{\mathbf{J}_3 + \int \sum_{j=1}^{n} (\lambda_j v_{j\bar{s}r}\xi^r + \lambda'_j v_{j\bar{s}'r}\xi^r)\,\mathrm{d}t + \int [\lambda_{n+1}(v_{n+1\,\bar{s}r}\xi^r \mp P_{n\bar{s},\,1s}\xi^s) + \lambda'_{n+1}(v_{n+1\,\bar{s}'r}\xi^r \mp P_{n\bar{s}'\,1s}\xi^s)]\,\mathrm{d}t\Big\} = 0. \tag{65}$$

The corresponding Euler–Lagrange equations comprise all the $2n+2$ constraints of the doubled chain, and along with them the system

$$\mathrm{D}\xi^r = -C_{rr'}\Big\{E_{r'r''}\xi^{r''} + E_{r's}\xi^s + \sum_{j=1}^{n+1}(\lambda_j v_{j\bar{s}r'} + \lambda'_j v_{j\bar{s}'r'})\Big\}, \tag{66}$$

$$0 = E_{sr}\xi^r + E_{ss'}\xi^{s'} \mp \lambda_{n+1}P_{n\bar{s},\,1s} \mp \lambda'_{n+1}P_{n\bar{s}',\,1s}. \tag{67}$$

The term $E_{ss'}$ in the last equation vanishes unless $n = 0$, in which case it takes over as a pivotal element subject to the previously stated rule of diagonal precedence. This special case can be included in the general scheme by understanding that $\sum_{j=1}^{0}$ means zero, and that $P_{0s,\,1s'}$ means $\mp E_{ss'}$.

Consider the primary equation of constraint (67) for the special case $s = \bar{s}$. The first (or, for $n = 0$, the first and second) member reduces to one of the two primary constraints of the doubled

chain, and therefore vanishes. The third term involves a diagonal pivotal element, and vanishes by hypothesis. The pivotal element in the last term is non-zero. Therefore

$$\lambda'_{n+1} = 0; \quad \text{similarly} \quad \lambda_{n+1} = 0. \tag{68}$$

Consider next (for $n \neq 0$) the time derivative of the primary constraint function $v_{1\bar{s}r}\xi^r$. One equation of the modified action principle asserts that it is zero, identically. But if we work it out by using (66) for $\mathrm{D}\xi^r$ and (52) for $\mathrm{D}v_{1\bar{s}r}$ we get

$$0 = \mathrm{D}(v_{1\bar{s}r}\xi^r) = v_{2\bar{s}r}\xi^r \mp P_{1\bar{s},\,1s}\xi^s - \sum_{j=1}^{n}(\lambda_j P_{1\bar{s},\,j\bar{s}} + \lambda'_j P_{1\bar{s}',\,j\bar{s}}). \tag{69}$$

In this expression the sum on j extends only up to n, by virtue of (68). The first term on the r.h.s. is a secondary constraint function, and therefore vanishes by virtue of (65). The second term and most of the terms in the summand involve vanishing pivotal elements (thanks to the sequential principle) and the sole survivor is that which contains λ'_n. Thus

$$\lambda'_n = 0; \quad \text{similarly} \quad \lambda_n = 0. \tag{70}$$

We go next to $\mathrm{D}(v_{2\bar{s}r}\xi^r)$, and so forth, and by using (57) and (58) prove successively that

$$\lambda'_j = \lambda_j = 0 \quad (j = n-1, n-2, \ldots, 2). \tag{71}$$

Finally, we consider the equation

$$0 = \mathrm{D}(v_{n\bar{s}r}\xi^r) = v_{n+1\,\bar{s}r}\xi^r \mp P_{n\bar{s},\,1s}\xi^s - \lambda_1 P_{n\bar{s},\,1\bar{s}} - \lambda'_1 P_{n\bar{s}',\,1\bar{s}}. \tag{72}$$

By referring to (53) we see that the first two terms on the r.h.s. combine together to give an $(n+1)$-ary constraint function (48), and therefore their sum vanishes by virtue of (65). The third term contains a vanishing diagonal pivotal element. The fourth yields

$$\lambda'_1 = 0; \quad \text{similarly} \quad \lambda_1 = 0. \tag{73}$$

Thus the Euler–Lagrange equations of the constrained Jacobi action principle (65) themselves ensure that all the Lagrange multipliers λ and λ' vanish on the orbits of (65), and hence (66) and (67) reduce precisely to the required Jacobi equations (43) and (44). Diagonal regular chains can be employed to effect prior eliminations in quite the same way, but with half as many Lagrange multipliers.

Invariance of the Lagrange bracket under reduction by prior elimination

We have shown explicitly that a correctly sequenced prior elimination of two of the ξ^s and $2n$ of the ξ^r in no way spoils the effectiveness of the Jacobi action principle (42), at least so far as concerns its equations of motion.

Let us consider now the kinetic term $\frac{1}{2}C_{rr'}\xi^{r'}\mathrm{D}\xi^r$, and see what becomes of it under these prior eliminations. Obviously it passes over into some non-canonical form $\frac{1}{2}\tilde{A}_{rr'}\xi^{r'}\mathrm{D}\xi^r$, in which the ranges of the summations have been decreased by $2n$. The new kinetic term is in fact obtained from the old one by making in the old one a total of $4n$ linear substitutions based on $2n$ of the constraints of the doubled $(n+1)$th-order chain, and upon their $2n$ first time derivatives, with subsequent rejection of any contributions not dependent upon the remaining $\mathrm{D}\xi^r$. The latter contributions involve factors $\mathrm{D}v$, and contribute only to the new potential term. Thus the $\mathrm{D}v$ are not involved in the reduction $C \to \tilde{A}$.

Consider next the Lagrange bracket (36), which in our special coordinates ξ is equal to

$$C_{rr'}\xi_2^{r'}\xi_1^r, \tag{74}$$

where ξ_1 and ξ_2 are any two solutions of the Jacobi equations (which, of course, entails that they obey all the constraints). It is evident that when we use the $2n$ constraints of our doubled chain to eliminate $2n$ of the $\xi_1{}^r$ and $2n$ of the $\xi_2{}^r$ from the above expression, we encounter a set of $4n$ substitutions having precisely the same structure as the set used when dealing with the kinetic term. Thus the Lagrange bracket (74) reduces to

$$\tilde{A}_{rr'}\,\xi_2{}^{r'}\,\xi_1{}^r, \tag{75}$$

where both the reduced matrix $\tilde{A}$, and the reduced range of summation, are the same as in the reduced kinetic term.

Thus the universal relation between the Lagrange bracket and the Jacobi Lagrange function is preserved under reduction by prior elimination, and the application of the standard definition (36) to the reduced Jacobi Lagrange function yields a construct equal in value to that of our original problem for all possible orbit displacements ξ_1 and ξ_2.

The theorem of rank for deterministic systems

By repeated canonization of A and correctly sequenced reduction, we may eventually reach a stage where, although the reduced Jacobi action still contains some ξ^s, the would-be pivotal elements (51) vanish for all n. In that case the constraint chains of the reduced system must terminate in self-repetition, as described in § 2. None of them ever involves a ξ^s, and the motion of the ξ^s, and hence also of some of the ξ^r, is indeterminate.

The alternative possibility is that we eventually reach a stage where the reduced Jacobi action contains no ξ^s and has for its equations of motion just the set (43), with the terms in ξ^s deleted. In this case it is clear both that a Jacobi orbit exists for every conceivable assignment of the residual ξ^r, and that the Jacobi orbit is fully deterministic. Also, because all conceivable residual ξ^r are allowed, and because $C_{rr'}$ is invertible, we have the condition of rank

$$\{C_{rr'}\,\xi_2{}^{r'}\,\xi_1{}^r = 0; \quad \forall \xi_2{}^{r'}\} \Rightarrow \{\xi_1{}^r = 0\}. \tag{76}$$

Here, in the anticlassical case, we appeal yet again to the quotient rule (8).

Since the Lagrange bracket in (76) has arisen from that of the action principle (1) by successive operations of linearization, canonization, and reduction, and since it is invariant under all these operations, we have

THEOREM 1. *For any action principle* (1) *whose Jacobi equations* (15) *are deterministic, the following property holds*:

$$\{\Delta_1 \times \Delta_2 = 0; \ \forall \Delta_2\} \Rightarrow \{\Delta_1 = 0\}, \tag{77}$$

where $\Delta_1 \times \Delta_2$ *is the bilinear differential form* (35), *and* Δ_1 *and* Δ_2 *are infinitesimal orbit displacements subject to all the constraints imposed by the Jacobi equations, and to no others.*

If the Lagrange function of the action principle (1) *is regular*, in the sense defined in §§ 1 and 4, *then we may substitute the name of Lagrange for that of Jacobi in both the places where Jacobi's name is used in the above statement.*

8. PROOF OF THE THEOREM OF RANK AND NULLITY FOR INDETERMINISTIC SYSTEMS

Irreducible systems and their indices of indeterminacy

Consider a reduced system in which there are $\mathscr{I}_1$ coordinates ξ^s remaining, and in which all the elements $E_{ss'}$ are zero. Up till now we have envisaged situations in which, by following sufficiently far along the $\mathscr{I}_1$ constraint chains, one finds somewhere a non-zero pivotal element $P_{n\bar{s},1\bar{s}'}$. It may

happen however that by the n-ary stage every constraint chain has reproduced constraints already found at an earlier stage, without having as yet yielded any non-zero pivotal elements. In this situation every chain may be cut short at the $(n-1)$-ary stage; no new constraints and no non-zero pivotal elements can ever arise by going further, and the system cannot be further reduced. The solutions of the Jacobi equations for such *irreducible systems* contain $\mathscr{I}_1$ arbitrary functions of t, since there are $\mathscr{I}_1$ velocities $D\xi^s$ which are not determined by (43) and (44) and their associated secondary, etc., constraints. The effect of these $\mathscr{I}_1$ arbitrary functions must perforce be fully and completely manifest in the temporal evolution of the $\mathrm{D}^r q^\alpha$. It is not possible for the arbitrary functions to affect only the motion of the auxiliary variables $\pi_{\alpha r}$, since on orbit these are fixed by (21). Therefore $\mathscr{I}_1$ is an invariant parameter of any regular system (1).

Suppose that an indeterministic and irreducible Jacobi system has altogether $\mathscr{I}-\mathscr{I}_1$ linearly independent primary, secondary, tertiary, ..., constraint vectors v. For reasons of an obvious nature we call $\mathscr{I}_1$ the *index of primary indeterminacy* and for reasons which will become apparent later we call $\mathscr{I}$ the *index of total indeterminacy*. Unlike $\mathscr{I}_1$, the index $\mathscr{I}$ depends upon the choice of coordinates used to represent the action principle (1). In particular, it depends upon the choice of M. It is only for convenience of presentation that we have chosen to work here with $M = 2N$. Thus $\mathscr{I}$ is not an invariant parameter of the system (1).

The simplest possible example of an irreducible classical Jacobi system (42) with a non-trivial secondary constraint has 4 coordinates ξ^r and 1 coordinate ξ^s, and if we choose these suitably to simplify their couplings, and label them in order from 1 to 5, so that s takes the value 5, then the relevant conditions to be imposed upon (42) take the form

$$\left.\begin{aligned} &E_{15} \neq 0, \quad E_{22} = 0, \quad E_{23} \neq 0, \quad E_{24} = E_{25} = 0, \\ &E_{35} = 0, \quad E_{44} = E_{45} = 0, \quad E_{55} = 0. \end{aligned}\right\} \tag{78}$$

This system has $\mathscr{I}_1 = 1$ and $\mathscr{I} = 3$, as the reader may readily verify.

In the terminology used by Dirac (1950, 1951, 1958, 1964*a*), the number $\mathscr{I}_1$ is the number of first class ϕ. The number $\mathscr{I}$ is close in concept to Dirac's concept of the total number of all possible independent first class linear combinations of first and second class ϕ with first and second class χ (not to be confused with the number of first class ϕ plus the number of first class χ, which on any unsophisticated reckoning is likely to come out smaller!). But it is difficult to say much more than this, because the two concepts are based on different methodologies, and neither of them has an invariant (i.e. coordinate–independent) significance.

The concept of initial value for indeterministic systems

Let us start the system (1) on the base orbit $\hat{q}$ at a fixed base time $\hat{t}$ so that, in the fully reduced description,

$$\xi^r(\hat{t}) = \xi^s(\hat{t}) = 0. \tag{79}$$

The equivalent conditions in the original description can be traced back through (28), (22), (18) and (17), and are

$$\left.\begin{aligned} &\Delta \mathrm{D}^r q^\alpha(\hat{t}) = 0 \quad (r = 0, 1, \ldots, 2N-1), \\ &\Delta \pi_{\alpha r}(\hat{t}) = 0 \quad (r = 0, 1, \ldots, 2N-2). \end{aligned}\right\} \tag{80}$$

It will be noted that the variables whose deviations are listed here are not necessarily the same as the variables $\mathrm{D}^r q^\alpha$ $(r = 0, 1, \ldots, M-1)$, which we previously took as an appropriate set of initial

values on which to start an orbit. There is no strict correlation between the two sets of variables, since the two integers M and $2N$, and indeed the very device of introducing these integers, are all to some extent arbitrary. Nevertheless, equations (20) provide a set of first-order equations of motion and a set of equations of constraint for the variables in (80), and therefore these variables can also serve in a perfectly adequate way as a set of initial values. From this it follows that for deterministic systems, where the orbit depends uniquely on its initial values, the two sets of variables are entirely equivalent in content, and therefore equivalent also in number after allowing for their respective sets of constraints. But for indeterministic systems there is no complete equivalence of this sort, since any variable q whose motion is not determined can provide in each case as many independent initial values as we like, merely by making M or N big enough. Thus the concept of what we mean by the initial values for an orbit loses its previous unique significance when we consider non-deterministic systems. For the present, therefore, we shall proceed on the arbitrary but convenient understanding that by initial values we mean the variables of (79) and (80) for some chosen N, compatible with the actual structure of L_1.

The Jacobi indeterminacy neighbourhood of an indeterministic orbit

Let us now propagate the variables of (79) forwards in time by some small finite amount, and then backwards again to $\hat{t}$, obeying the Jacobi equations all the while. Since the $D\xi^s$ are arbitrary, and can be chosen independently on the outward and return journeys, the orbit may wander away from the chosen base orbit, and the point of return at $\hat{t}$ may deviate *finitely* in ξ, i.e. infinitesimally in the $\mathrm{D}^r q^\alpha$ and the $\pi_{\alpha r}$, from the starting point. The finite deviations of ξ obtained in this way can be compounded linearly, since the Jacobi equations are linear. Amongst them there will obviously be found a linear subset spanned by the set of $\mathscr{I}_1$ independent directions $X_{s'}$ whose components are given by

$$X_{s'}{}^r = 0, \quad X_{s'}{}^s = \Upsilon \delta_{s'}{}^s \quad (s' = 1, 2, \ldots, \mathscr{I}_1), \tag{81}$$

where Υ is any non-zero constant, anticommutative in the anticlassical case.

These considerations lead us to introduce the coordinate-dependent concept of *Jacobi indeterminacy neighbourhood*, as the set of all orbit points at $\hat{t}$ which, lying in the differential neighbourhood of the base point at $\hat{t}$, can be reached from that base point by repeated small forward and backward integrations of the Jacobi equations (43) and (44).

To study this concept it is advantageous to place t and the ξ on the same footing, and so consider instead a *Jacobi motion neighbourhood*, whose section at $t = \hat{t}$ will provide the Jacobi indeterminacy neighbourhood. It is advantageous also to extend definitions (81) to a general base point in ξ space, and to a general time, and to include with them the extra specifications

$$X_{s'}{}^t = 0 \quad (s' = 1, 2, \ldots, \mathscr{I}_1). \tag{82}$$

The $X_{s'}$ then become direction fields in (ξ, t) space. And since we are studying the motion in this space we adjoin to the $\mathscr{I}_1$ fields of direction $X_{s'}$ one further field X_t whose components, by (43) and (44), are

$$\left.\begin{aligned} X_t{}^r &= -C_{rr'}(E_{r'r''}\xi^{r''} + E_{r's}\xi^s), \\ X_t{}^s &= 0, \\ X_t{}^t &= 1. \end{aligned}\right\} \tag{83}$$

We can now appeal to Frobenius's integrability theorem (Schouten & Kulk 1949*b*, see also § 9 of the present work), which asserts that if a one dimensional path of finite length (and homo-

topic to the zero path) be constructed along a set of direction fields $X_j{}^a$ in a space x^a, then all the Lie commutators

$$X_{[j,\,k]}{}^a \equiv X_{k,\,b}^{a}\, X_j{}^b - X_{j,\,b}^{a}\, X_k{}^b \tag{84}$$

will appear in the differential motion neighbourhood as soon as one includes terms of second order in the length of the path, and that, in higher orders, all possible repeated Lie commutators will appear, but no terms of any other description.

The only non-zero unrepeated Lie commutators (84) for our problem are the $X_{[s',\,t]}$, which represent lack of determinacy in the accelerations $\mathrm{D}^2\xi$. Using (49), we find that they are closely related to the primary constraint vectors of our irreducible system

$$\left.\begin{aligned} X_{[s',\,t]}{}^r &= -\Upsilon C_{rr'} E_{r's'} = \Upsilon v_{1s'r'}\, C_{r'r}, \\ X_{[s',\,t]}{}^s &= X_{[s',\,t]}{}^t = 0. \end{aligned}\right\} \tag{85}$$

Lie commutators between (85) vanish trivially, as do those between (85) and (81). So we go to the once-repeated commutators which account for lack of determinacy in the $\mathrm{D}^3\xi$, and we find, by (52), that

$$\left.\begin{aligned} X_{[[s',\,t],\,t]}{}^r &= -\Upsilon v_{2s'r'}\, C_{r'r}, \\ X_{[[s',\,t],\,t]}{}^s &= X_{[[s',\,t],\,t]}{}^t = 0. \end{aligned}\right\} \tag{86}$$

Proceeding in this way, *all* the residual constraints $\times\, C$ turn up as repeated Lie commutators, and once they have all turned up, any further Lie commutation produces only linear combinations of what we have already. For the relevant equation which carries us from one stage to the next is always (52), and this, as we noted at the beginning of this section, terminates by repetition for any irreducible system.

These arguments based on Lie commutation prove that the differentials of total indeterminacy of the Jacobi equations are spanned by the $\mathscr{I}_1$ independent directions (81) and the $\mathscr{I}-\mathscr{I}_1$ independent members of the series (85), (86), etc. But the Jacobi equations are linear in the ξ, and therefore the same applies to the finite indeterminacies of the ξ, and hence to the total indeterminacy differentials

$$\left.\begin{aligned} \Delta D^r q^\alpha &\equiv \epsilon \mathrm{D}^r \eta^\alpha \quad (r = 0, 1, \ldots, 2N-1), \\ \Delta \pi_{\alpha r} &\equiv \{\partial \pi_{\alpha r}/\partial(\mathrm{D}^{r'} q^{\alpha'})\}\, \epsilon \mathrm{D}^{r'} \eta^{\alpha'} \quad (r = 0, 1, \ldots, 2N-2), \end{aligned}\right\} \tag{87}$$

of the Jacobi equations (15) of the original system (1). Thus, every variable ξ^s of the residual irreducible Jacobi system is associated both with an indeterminacy differential (87) of the initial values of the original Jacobi system (15) and with an arbitrary function of t in the solution of these equations, and every constraint of the residual irreducible system is associated with an indeterminacy differential (87) of (15), and there are no indeterminacy differentials of the original Jacobi system (15) independent of these.

The theorem of rank and nullity for indeterministic systems

Each Jacobi indeterminacy differential $\epsilon X_{s'}$, $\epsilon X_{[s',\,t]}$, ... is a possible instantaneous orbit displacement of the Jacobi orbit, and as such may figure as an argument in the bilinear differential form (35) or (33). We shall now use the Lagrange bracket, with its properties of invariance under linearization, canonization, and reduction, to give an invariant characterization of the X.

Consider the following invariantly stated problem

$$\left.\begin{aligned} &\textit{ϵX and Δ are infinitesimal orbit displacements, subject to the constraints imposed by the Jacobi} \\ &\textit{equations of (1), and no others, $\Delta_1 \times \Delta_2$ is the bilinear differential form (35), and} \\ &\epsilon X \times \Delta = 0 \quad (\forall \Delta). \end{aligned}\right\} \tag{88}$$

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We shall solve this problem by writing it in terms of the Jacobi coordinates of the residual irreducible system. In that system the respective statements are

$$v_{1sr}X^r = 0, \quad v_{2sr}X^r = 0, \ldots, \tag{89}$$

$$X^r C_{rr'}\Delta^{r'} = 0; \quad \forall\Delta \text{ such that} \tag{90}$$

$$v_{1sr}\Delta^r = 0, \quad v_{2sr}\Delta^r = 0, \ldots. \tag{91}$$

In these equations we must understand that a complete set of $\mathscr{I} - \mathscr{I}_1$ linearly independent v are employed. Then equations (91) constitute $\mathscr{I} - \mathscr{I}_1$ distinct constraints on Δ^r, and if r ranges over $\mathscr{R}$ values, the solutions Δ^r of (91) constitute a linear space of dimension $\mathscr{R} - (\mathscr{I} - \mathscr{I}_1)$. Equations (90) therefore impose on $X^r C_{rr'}$ a total of $\mathscr{R} - (\mathscr{I} - \mathscr{I}_1)$ independent conditions, and those X^r which satisfy (90) constitute (since C is invertible) a linear space of dimension

$$\mathscr{R} - \{\mathscr{R} - (\mathscr{I} - \mathscr{I}_1)\} = \mathscr{I} - \mathscr{I}_1.$$

Let the symbol λ stand generically for an indeterminate quantity. Clearly (91) imply that (90) are satisfied by

$$X^r C_{rr'} = \lambda^{1s} v_{1sr'} - \lambda^{2s} v_{2sr'} + \ldots, \tag{92}$$

and since this solution already contains $\mathscr{I} - \mathscr{I}_1$ independent parameters it must be the general solution of the system (90) with (91), so far as concerns the X^r. Equations (89) impose no further conditions, because of the vanishing of all the residual scalar products (51) and (57).

The components X^s of X are not restricted in any way by (89)–(91). Thus, by comparing (92) with (81), (85) and (86) we get, as the most general solution of (89)–(91), the $\mathscr{I}$-parameter family

$$\epsilon X = \lambda^s \epsilon X_s + \lambda^{1s} \epsilon X_{[s,\,t]} + \lambda^{2s} \epsilon X_{[[s,\,t],\,t]} + \ldots. \tag{93}$$

But this is the most general possible indeterminacy differential of the Jacobi equations.

In §9 we shall prove that for regular variational problems the Lagrange indeterminacy differentials and the Jacobi indeterminacy differentials span the same linear space. This result is by no means obvious. Anticipating it, we can now enunciate

THEOREM 2. *Every differential ϵX of total indeterminacy of the Jacobi equations* (15) *of* (1) *satisfies the statements* (88), *and* (88) *have no non-zero solutions X other than these.*

If the Lagrange function of the variational principle (1) *is regular*, in the sense defined in §§1 and 4, *then we may substitute the name of Lagrange for that of Jacobi both in the preceding sentence and in the statement* (88) *referred to therein.*

9. THE THEORY OF INDETERMINACY IN THE LARGE – COMPLETION OF THE PROOF OF THE THEOREM OF RANK AND NULLITY

Space–time notation

In §2 it was shown that the orbits lie in and fill a differentiable manifold which, in the coordinate system of §5, is defined by the simultaneous vanishing of a set of differentiable constraint functions $\Psi_e(t, q^a)$ $(e = 1, 2, \ldots)$. In the present section we shall show that the orbits mark out a continuum of mutually exclusive submanifolds in the orbit manifold and in its fixed-time sections.

The bulk of our discussion can best be formulated in a space–time symmetric notation, to

which end we introduce a running index A whose values comprise those of a and the one extra value t. In order to accommodate the various signs appropriate to anticlassical systems we introduce also a sign symbol (AB) such that:

$$\left.\begin{array}{ll}\text{for anticlassical systems} & (tt) = (at) = (ta) = -(ab) = 1,\\ \text{for classical systems} & (AB) = 1.\end{array}\right\} \tag{94}$$

When using this symbol we adopt the rule that the indices A and B of (AB) are to be excluded from the provisions of the summation convention.

On this understanding equations (24) can be extended to

$$\mathsf{A}_{AB} = p_{A,B} - (AB)p_{B,A} = -(AB)\,\mathsf{A}_{BA}, \tag{95}$$

where the new entity A_{tt} introduced hereby evidently obeys always

$$\mathsf{A}_{tt} = 0. \tag{96}$$

The Lagrange equations (25) and the resulting energy equation (26) can be written jointly in the space–time symmetric form

$$\mathsf{A}_{AB}\mathrm{D}q^B = 0, \tag{97}$$

to which without prejudice we may adjoin the complete set of equations of constraint

$$\Psi_e = 0, \quad \Psi_{e,A}\mathrm{D}q^A = 0 \quad (e = 1, 2, \ldots). \tag{98}$$

These equations are consequences of (97), and in their derivation from (97) it is essential to take into account the special nature of the displacement D, which symbolizes always a movement in which $q^t \equiv t$ is changing (cf. the discussion around equation (27)).

In exploiting the differential members of (98) we shall always assume that the Ψ_e have been written in such a form that the rank of the matrix of derivatives $\Psi_{e,a}$ is equal to the number of independent Ψ_e.

The extended Lagrange bracket

By suitably modifying the formulae (33) it is possible to express the Lagrange bracket in terms of displacements Δ with non-zero time components $\Delta q^t \equiv \Delta t$. In this extended context we must understand each Δ to represent a displacement from one orbit to another infinitesimally close orbit, in which the two orbits involved are held fixed, while the two points between which the separation is Δ are allowed to move at will by any infinitesimal amounts along their respective trajectories. The infinitesimal displacement Δ so defined is arbitrary to within the transformation

$$\Delta q^A \to \Delta q^A + \epsilon \mathrm{D}q^A, \tag{99}$$

where ϵ is any infinitesimal multiplier, and it obeys the Lagrange equations of constraint in the space–time form

$$\Psi_{e,A}\Delta q^A = 0. \tag{100}$$

From (97) and (95) and the evenness–oddness properties of the q^A it follows that the Lagrange bracket (33) can be expressed in terms of the extended spatio–temporal Δ by the formula

$$\Delta_1 \times \Delta_2 = \mathsf{A}_{AB}\Delta_2 q^B \Delta_1 q^A. \tag{101}$$

The property of closure of the bilinear form

The rule (11) for second derivatives goes over to

$$F_{,BA} \equiv \frac{\partial}{\partial q^A}\left(\frac{\partial F}{\partial q^B}\right) = (AB)\frac{\partial}{\partial q^B}\left(\frac{\partial F}{\partial q^A}\right) \equiv (AB)\,F_{,AB} \tag{102}$$

in the space–time symmetric notation. From this and (95) it is an elementary matter to show that the matrix A_{AB} has a property of *closure*

$$(AC)\,\mathsf{A}_{AB,C} + (CB)\,\mathsf{A}_{CA,B} + (BA)\,\mathsf{A}_{BC,A} = 0. \tag{103}$$

This property will prove to be significant in the theory of the submanifolds of the orbit manifolds of singular systems (§ 10), in the theory of the inverse problem (§ 11), and in the theory of canonical reduction (§ 12).

Differential properties of the vectors of Lagrange motion and the vectors of primary Lagrange indeterminacy

The constraints (98) are supposed to be a complete set, constructed from (97) by the methods of § 2. From the theory of that section it follows that any solution $X_{(1)}{}^A$ of the simultaneous linear system

$$\mathsf{A}_{AB}X_{(1)}{}^B = 0, \quad \Psi_{e,B}X_{(1)}{}^B = 0, \tag{104}$$

with $X_{(1)}{}^t$ even and $X_{(1)}{}^a$ odd in the anticlassical case, represents a possible Lagrange motion or segment of orbit of the Lagrange equations if $X_{(1)}{}^t \neq 0$, and conversely, and that any solution subject to the extra linear condition

$$X_{(1)}{}^t = 0 \tag{105}$$

represents a primary indeterminacy of the Lagrange equations, and conversely.

Thus the vectors of Lagrange motion and the vectors of primary Lagrange indeterminacy span the right null space of a differentiable rectangular matrix whose elements are A_{AB} and $\Psi_{e,B}$, while the vectors of primary Lagrange indeterminacy by themselves span the right null space of a differentiable rectangular matrix whose elements are A_{AB}, $\Psi_{e,B}$ and δ_{tB}.

Now any right null vector of a rectangular matrix of rank R can be expanded linearly over a complete set of *rational right null vectors*, these being right null vectors X whose various components X^A are signed subdeterminants of dimension R. By such an expansion any null vector of a differentiable and single-valued rectangular matrix can be continued differentiably as a single-valued null vector throughout any domain in which the rank is bounded by R. Applying this to the problem in hand, it becomes clear that the various vectors $X_{(1)}$ of the Lagrange equations can be taken to be differentiable and single-valued fields extending globally over the orbit manifold in q^A-space. Local degeneration of any particular global rational null field $X_{(1)}$ can without exception be accommodated by using a sufficiently redundant set of global $X_{(1)}$, so that wherever one fails another is already available to take over.

Differential properties of the secondary, etc., vectors of Lagrange indeterminacy

We have already seen how to formulate the concepts of differential neighbourhood of total motion and differential neighbourhood of total indeterminacy for the Jacobi equations, by processes of repeated Lie commutation among the Jacobi $X_{(1)}$. The same concepts and methods can

be extended to the Lagrange equations, by constructing all possible repeated Lie commutators of the various rational solutions $X_{(1)}$ of the system (104). Clearly the vectors X so constructed constitute a set of differentiable and global fields.

Closure property of the vectors X of Lagrange motion and Lagrange indeterminacy

The process of repeated Lie commutation between all the $X_{(1)}{}^A$ must eventually terminate in redundancy, if only because the number of linearly independent X^A cannot exceed the number of dimensions of the space $\{q^A\}$. Thus the X^A are closed under Lie commutation, and by Frobenius's integrability theorem (Schouten & Kulk 1949*b*) they generate integral submanifolds, with dimension equal to the number of linearly independent X.

Motion and gauge submanifolds

If therefore we take a base point $\hat{q}^A$ in the orbit manifold, and integrate Lagrange's equations (97) from this point, allowing for all primary indeterminacy, and allowing also reversals of the temporal direction of integration, then the set of all points q^A encountered in all possible such integrations constitutes a submanifold of the orbit manifold. As indicated above, the dimension of this submanifold is equal to the number of linearly independent X, which is at least unity, and its differential neighbourhoods are spanned by the local X.

These submanifolds will be called *motion submanifolds*. It is clear that if $\hat{\hat{q}}$ lies in the motion submanifold of $\hat{q}$, then the motion submanifold of $\hat{\hat{q}}$ includes $\hat{q}$, and is the same as that of $\hat{q}$.

The section at time $t = \hat{t}$ of the motion submanifold of a point $\hat{q}$ is called in physical applications the *gauge submanifold* of $\hat{q}$. The gauge submanifolds are generated by those X which obey the condition $X^t = 0$, and the number of such X which are linearly independent is always less by unity than the total number of linearly independent X.

Obviously the motion submanifolds and the gauge submanifolds can have no boundaries, save at points where the Lagrange equations become singular or degenerate. They may however be spacewise multiply connected, as may the orbit manifold itself.

Gauge invariants, integrals of the motion, and constants of the motion

If throughout the orbit manifold a function $G(q^A)$ obeys the closed differential system

$$G_{,a} X^a = 0 \quad (\forall X \text{ such that } X^t = 0), \tag{106}$$

it is said to be a *gauge invariant* function. The rate of change DG of any gauge invariant function along any orbit is fully determined for all time, even though the evolution of the orbit itself is not. Conversely, if the rate of change of a function G along any orbit is for all time fully insensitive to the indeterministic wanderings of the orbit, and if this holds for all orbits, then the function G in question is a gauge invariant function.

If a gauge invariant obeys at all points of the orbit manifold the stronger conditions

$$G_{,A} X^A = 0 \quad (\forall X), \tag{107}$$

it is said to be an *integral of the motion*.

Any integral of the motion obeying at all points of the orbit manifold the further condition

$$G_{,t} = 0 \tag{108}$$

is said to be a *constant of the motion*, or a conserved quantity.

Clearly concept (107) is invariant under any conceivable change of coordinates, including even changes which affect the integrity of t. Concept (106) is invariant under all changes for which t is scalar. Concept (108) is not generally an intrinsic concept. In practice it is useful only for systems for which $\partial L_1/\partial t \equiv 0$, in which cases it becomes intrinsic. All three concepts apply equally in the coordinates q^A and in the coordinates (t, q^α) of L_1.

Relation between indeterminacy differentials realizable by infinitesimal and by finite deviation of the orbit–completion of the proof of the theorem of rank and nullity

The concepts of motion and gauge submanifold introduced above depend upon the use of many finitely deviating orbits. This is to be contrasted with the apparently less ambitious account of indeterminacy rendered by the Jacobi equations, where all the deviations considered appear as infinitesimal deviations of a single finite orbit segment when mapped into the descriptive scheme of the Lagrange equations. Since an account based on the infinitesimal deviations of a single orbit is not conceptually adequate for a full discussion of what one means by the indeterminacy of many finitely deviating orbits, one might expect that the inclusion of finite deviations and many orbits would increase the number of dimensions of the indeterminacy differential neighbourhood. Rather surprisingly, however, a more careful analysis of the implications of general Lie theory reveals no such increase for regular Lagrangian systems.

The general Lagrange orbit can be regarded as a sequence of infinitesimal displacements, each proportional to a local vector of the form

$$Y^A \equiv X_{(1t)}{}^A + \sum_j \lambda^j X_{(1j)}{}^A, \tag{109}$$

where $X_{(1t)}$ is one of the $X_{(1)}$, having unit temporal component, the λ^j are coefficients varying arbitrarily along the orbit, and the $X_{(1j)}$ are a complete set for those $X_{(1)}$ which have zero temporal component. Clearly each $X_{(1j)}$ appears in the differential indeterminacy neighbourhood of any infinitesimally deviating orbit running through the point of application. Consider therefore two infinitesimally close orbits, which start off at the same point, and which subsequently deviate along some $X_{(1k)}$. Let the two orbit points be propagated through a further small time interval by following the field (109) for some chosen and fixed values of the λ^j. The direction followed will depend upon the λ^j. The deviation Δ^A, having already attained a value proportional to $X_{(1k)}{}^A$, is also propagated by (109), and in this propagation suffers a change proportional to $Y^A{}_{,B} X_{(1k)}{}^B$. The local components of $X_{(1k)}{}^A$ change meanwhile by an amount proportional to $X_{(1k),}{}^A{}_B Y^B$. Thus at the end the deviation Δ^A differs from the local $X_{(1k)}{}^A$ by an amount proportional to

$$Y^A{}_{,B} X_{(1k)}{}^B - X_{(1k)}{}^A{}_{,B} Y^B, \tag{110}$$

which is the Lie commutator $[X_{(1k)}, Y]^A$. Since the λ are at our disposal, we can thus construct various orbits, some showing one Lie commutator and some another, at their respective terminals. By following these orbits backwards similar Lie commutator deviations can be induced at their common starting point. Thus all first-order Lie commutators $X_{[(1k),(1t)]}$ and all first-order Lie commutators $X_{[(1k),(1j)]}$ can be independently realized at a common point by *infinitesimal deviation* of various suitably chosen base orbits of (109) through that point.

This argument extends to Lie commutators of higher order. Every multiply repeated Lie commutator of the primary Lie field system (109) at a point $\hat{q}$ can already be realized by infinitesimal deviation of some appropriately chosen base orbit of (109) through $\hat{q}$. The import of

Frobenius's integrability theorem is therefore that finite deviations yield nothing more, once all base orbits are taken into account.

For regular Lagrangian systems the concept of infinitesimal deviation of a base orbit leads through the Jacobi equations and the associated theorem of rank and nullity to the set of indeterminacy differentials X of equations (88). But equations (88) are structurally invariant. They depend upon the coordinates $\hat{q}$ of the point of application, but make no reference whatever to the choice of base orbit through that point. Therefore we have the following very remarkable result

THEOREM 3. *For regular Lagrangian systems all base orbits realize the full indeterminacy even in their infinitesimal deviation, and the Lagrange X are the same in number and content as the Jacobi X.*

This does not of course imply that the Lagrange X and the Jacobi X of a regular problem are individually equal. The Lagrange vectors usually depend non-trivially upon the q^a, while the Jacobi vectors do not depend upon the η^a, and therefore the detailed Lie commutator structure can differ very considerably between the two sets of X. Nevertheless, the two sets span the same linear vector space for any regular Lagrangian system, provided only that the implicit properties of differentiability to finite order are satisfied.

The Lagrangian version of theorem 2 on rank and nullity, presented without proof at the end of §8, *is hereby established.*

The application of these ideas to singular Lagrangian systems is hampered by the fact that the constraint members (100) of the Lagrange equations may be stronger than the constraints of the Jacobi equations, and also by the fact that the form of the latter constraints may depend not only upon the base point $\hat{q}$, but also upon the indeterminate velocities, accelerations, etc., with which the chosen base orbit passes through $\hat{q}$. Different base orbits may yield different Jacobi X at the same base point $\hat{q}$, by virtue of the fact that the content of (88) depends upon the constraints of the Jacobi system.

On the other hand, the Jacobi equations are always obeyed, and although for singular systems they do not completely delimit the range of possible orbit deviations, they do provide valid limitations on the deviations which can actually occur.

From this and the foregoing analysis it therefore follows that *for singular Lagrangian systems the linear space of all Lagrange X^a at $\hat{q}$ is a subspace of the space spanned by the Jacobi X^a of all possible base orbits through $\hat{q}$*. In words which are plainer but less exact, the differential Lagrange indeterminacy of a singular Lagrangian system can never exceed the aggregated Jacobi indeterminacy, and may be less (as in the trivial example $L_1 = (q)^3$). Perhaps it is always less.

Invariance of the Lagrange bracket under displacements of motion and gauge at fixed base point

Consider the Lagrange bracket in its original form (33). In concept it involves three infinitesimally close orbits, one of which may be taken as base orbit. It has just been shown that by suitable choice of base orbit any required indeterminacy vector X of the Lagrange equations can be realized by infinitesimal deviation of one of the other two orbits along a finite segment of its length. Since the Lagrange bracket is conserved under all natural motions its value with the infinitesimal deviation present and its value without must be equal, since the starting values concide. So we have

$$\Delta_1 \times \Delta_2 = \Delta_1 \times (\Delta_2 + \epsilon X) \quad (\forall \Delta_1). \tag{111}$$

Bringing together this and the earlier results of the present section we can now enunciate a fourth theorem.

THEOREM 4. *For any variational problem* (1), *whether regular or singular, the vectors* X ($X^t = 0$) *of primary, secondary, tertiary, etc. Lagrange indeterminacy enjoy the following three properties*:

(i) *Let* Δ *be an infinitesimal orbit-to-orbit displacement, subject to the constraints imposed by the Lagrange equations, and no others. Then every Lagrange indeterminacy vector* X *satisfies the invariant nullity conditions*

$$\Delta \times X = 0 \quad (\forall \Delta), \tag{112}$$

where $\Delta_1 \times \Delta_2$ *denotes the bilinear differential form* (35).

(ii) *The vectors* X *are closed under Lie commutation, and generate fixed-time submanifolds of the orbit manifold, called gauge submanifolds. Their dimension is equal to the number of linearly independent* X.

(iii) *The motion through time carries gauge submanifolds into gauge submanifolds, and by this process generates motion submanifolds, whose dimension exceeds the number of linearly independent* X *by unity.*

The structure of (112) indicates the possibility of an alternative (and very tortuous) derivation, based upon double application of theorem 2. However, the simpler derivation given above makes it clear that (112) is logically independent of the provisions of that theorem. It also makes it clear that the virtue of that theorem is that it permits certain converse statements to be made.

The property of gauge and motion invariance (112) can also be proved by a direct calculation, which will be described in § 10. The calculation reveals that (112) depends in an essential way upon the property of closure (103).

In the derivation of (112) we have used the fixed-time Lagrange bracket (33) or (35), rather than its extended counterpart (101). But since (33) and (101) are equal we can understand (112) to apply to the extended bracket also. In this extended context Δ is any infinitesimal space–time displacement in the orbit manifold, and ϵX is any infinitesimal space–time displacement in the local motion submanifold.

Invariance of the Lagrange bracket under motion and gauge displacements of the base point

Since every motion and gauge displacement of the base point of the Lagrange bracket can be realized by an extended natural motion, under which the Lagrange bracket is conserved, it follows that (112) by itself gives only a partial expression to the full properties of invariance. The remaining property can be expressed in differential form by the further condition

$$\mathsf{A}_{AB,C} X^C \Delta_1{}^B \Delta_2{}^A + \mathsf{A}_{AB} X^B{}_{,C} \Delta_1{}^C \Delta_2{}^A + \mathsf{A}_{AB} \Delta_1{}^B X^A{}_{,C} \Delta_2{}^C = 0 \quad (\forall \Delta_1, \Delta_2), \tag{113}$$

where X is any differentiable field of Lagrange motion and indeterminacy.

This condition is independent of (112), to the extent that it cannot be derived from (112) save by use of the property of closure (103). It can also be derived directly from the closure property, by a method set out in § 10.

An invariant statement of Hamilton's equations

The property (112) tells us about the differential indeterminacy neighbourhood of the Lagrange motion. But in its extended form it also tells us something about the differential description of the motion itself. The presence of the first derivatives of the energy function $-p_t$ inside the Lagrange bracket (101) indicates that we deal here with a non-canonical but invariant generalization of Hamilton's canonical equations. We shall now study this generalization, and find out to what extent it yields a proper description both of the dynamics and of its indeterminacy.

The implications of the generalized Hamilton equations – motion modulo the gauge group

Every vector $X_{(1)}$ of the Lagrangian motion (104) satisfies the space-time version of the Lagrange bracket equation (112), and has non-zero temporal component. Let X denote another

vector of displacement in the orbit manifold, having equal temporal component, and suppose X also satisfies (112). Then $X-X_{(1)}$ is a spatial vector of displacement in the orbit manifold, obeying (112).

For regular systems we can now at once apply the theorem of rank and nullity in its Lagrangian version (§ 8) to show that $X-X_{(1)}$ belongs to the indeterminacy set or gauge set of the Lagrange equations. Evidently (112) and the constraints provide between them a full statement of the law of motion modulo the gauge group, for all regular systems.

For singular systems the above argument fails at several points, which will be evident to the reader. Sometimes, by accident, the inadequacies of the Jacobi equations may be compensated by the feature that in (112) it is the stronger Lagrange constraints that are to be used. This happens, for example, for the singular and indeterministic system

$$L_1 = q^1(q^2)^2, \tag{114}$$

for which (112) provides a complete statement of the motion modulo the gauge group. But it may alternatively transpire, as in the singular and indeterministic example

$$L_1 = q^1(\mathrm{D}q^2)^2, \tag{115}$$

that (112) allows more freedom of motion than is present in the gauge group. For the generic singular system the Hamilton equations (112) must therefore be reckoned as incomplete even at the level of description modulo the gauge group.

The index of determinacy

The number $\mathscr{I}_1$ of arbitrary functions of t in the solution of Lagrange's equations is coordinate independent but, as already noted, the dimensionality $\mathscr{I}$ of the gauge submanifolds is not. We can however introduce an invariant integer complementary to $\mathscr{I}$, by appealing to the above concept of motion modulo the gauge group. Suppose the system (1) admits a causal description in terms of $M\nu$ initial values $\mathrm{D}^r q^\alpha$ ($r = 0, \ldots, M-1; \alpha = 1, \ldots, \nu$). Suppose that in this description the number of independent constraints is $\mathscr{C}(M, \nu)$, and suppose that in the same description the number of independent Lagrange indeterminacy vectors is $\mathscr{I}(M, \nu)$. Then the orbit manifold is composed of layers of dimension $\mathscr{I}(M, \nu) + 1$, stacked one upon the other. These layers require precisely

$$\mathscr{D} \equiv M\nu - \mathscr{C}(M, \nu) - \mathscr{I}(M, \nu) \tag{116}$$

integrals (107) of the motion to distinguish them from each other; alternatively we may say that Lagrange's equations determine the rates of change of precisely $\mathscr{D}$ kinematically independent gauge invariants (106). The number $\mathscr{D}$ may be called the index of determinacy of the system (1). Clearly it is invariant under all types of coordinate transformation, unlike $\mathscr{I}$.

Evenness of the index of determinacy for regular classical systems

In the case of regular systems we can also write

$$\mathscr{D} = \mathscr{R}(M, \nu) - 2\{\mathscr{I}(M, \nu) - \mathscr{I}_1\}, \tag{117}$$

where $\mathscr{R}(M, \nu)$ and $\mathscr{I}(M, \nu) - \mathscr{I}_1$ denote respectively the number of residual ξ^r and the number of independent residual constraints in the reduced Jacobi system of § 8. Now in the classical case $\mathscr{R}(M, \nu)$ is always even. So we infer

THEOREM 5. *For regular classical Lagrangian systems the index of determinacy $\mathscr{D}$ is always even. But for anticlassical systems $\mathscr{D}$ may be even or odd, without restriction.*

10. Applications to Dirac's generalized Hamiltonian dynamics

Dirac's gauge concept

Consider again the Lagrange bracket equation (112), and its implications for displacements at fixed time. The equation states that every Lagrange gauge displacement X^a is a null vector of a closed (103) bilinear form $A_{ab}\Delta_2 q^b \Delta_1 q^a$ on the spatial section of the orbit manifold. In his fundamental papers on generalized Hamiltonian dynamics, Dirac (1950, 1951, 1958) introduced an alternative concept of gauge displacement which, although rather different in outward appearance, can be shown nevertheless to comprise the set of all null vectors of a closed bilinear form. Now for regular systems we can apply the Lagrangian version of the theorem of rank and nullity (§ 8), which tells us that the only spatial solutions of (112) are the vectors of Lagrange indeterminacy. For such systems therefore the Dirac gauge concept agrees with that developed in the present work. But the example (115) clearly demonstrates that for singular systems the dimension of Dirac's gauge submanifold can exceed that which arises from Lagrange indeterminacy, in which case the two concepts are inequivalent.

Dirac's conjecture on indeterminacy and gauge

In a later work Dirac (1964*c*) drew attention to a suggestive connexion between gauge displacements, repeated Poisson brackets, and indeterminacy, and put forward the conjecture that his gauge displacements (i.e. the null vectors of the bilinear form) and the displacements of total (primary, secondary, ...) Lagrange indeterminacy are one and the same.

This conjecture is now seen to be correct for all regular systems, and since every quadratic system is *de facto* regular, it is correct for most, and probably all, of the systems encountered in physical applications of action principles. But for many singular systems, of which the simplest is the cubic system (115), the conjecture fails.

The mechanism of failure brings with it in Dirac's formalism a pretty paradox, which is worthy of mention and illustration here.

The primary constraints of Dirac's methodology are denoted by the letter ϕ, and those of them which are *first class* (Dirac 1964*b*) generate the primary indeterminacies. The secondary, tertiary, etc., constraints are denoted by the letter χ. The χ all arise by a process of repeated Poisson bracket formation, which serves in Dirac's formalism as Lie bracket formation serves in the present method. Dirac's conjecture is to the effect that if we put all the ϕ and all the χ together, then every first class linear combination that can be drawn from the total fund of constraints so afforded will generate an infinitesimal motion of the total indeterminacy group of the system, and vice versa. In the case of the illustrative example (115) the Dirac formalism yields one first class $\phi \equiv p_1$, and this generates correctly the evident indeterminacy of motion of q^1. There arises also one further constraint function $\chi \equiv p_2$, which is also of first class, and which generates displacements of q^2 in Dirac's gauge group. It is these latter displacements which negate the Dirac conjecture, since the motion of q^2 is deterministic ($\mathrm{D}q^2 = 0$).

The constraint $p_2 = 0$ comes about through the equation to zero of a Poisson bracket between the primary indeterminacy generator ϕ and the generator p_t of temporal displacement. It is here that the paradox appears. For the theory of Lie commutation assures us without doubt that this Poisson bracket can be taken as a second generator of indeterminacy, and yet at the same time we also know that for (115) there cannot be such a second generator. The paradox is resolved by noting that the Poisson bracket in question is proportional to the square of p_2. Thus, while the

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equation to zero of the Poisson bracket yields correctly the constraint $p_2 = 0$, the bracket itself is impotent as a generator, because of the extra vanishing factor. This unexpected behaviour must surely be typical of any singular system for which the Dirac gauge group exceeds the gauge group of the Lagrange equations.

Integrability of the null vectors of a closed bilinear form on a manifold

In § 9 we used the method of repeated Lie commutation to show that the gauge displacements of the Lagrange equations are *de facto* integrable. These considerations, supplemented by the theorem of rank and nullity, clearly suffice to demonstrate that Dirac's displacements of gauge are integrable for all regular systems. But they do not, in general, cover the latter displacements for singular systems. In his first paper on generalized Hamiltonian dynamics, however, Dirac (1950) used the powerful method of Lie commutation to pursue a course different from that followed in the foregoing, and with the help of this method proved that his displacements are integrable in general. This salutary property is vital in Dirac's theory of Poisson brackets (Dirac 1950, 1951, 1958, 1964*a*).

The property of integrability of the null vectors of a closed bilinear form in $\mathbb{R}_n$ has been known for a long time. It was first discovered, almost accidentally one might say, in the study of Pfaff's problem, in which context it has been proved by a wide variety of methods of varying complexity and varying intrinsic merit (Goursat 1922*b*; Thomas 1937; Cartan 1946; Schouten & Kulk 1949*a*; Slebodzinski 1963; Carathéodory 1965). Of all these methods, that which uses Lie commutators (Schouten & Kulk 1949*b*; Dirac 1950) is the most efficient and the most geometric by far. Yet it receives scarcely a mention in the expository literature of the Pfaff problem. That this is so no doubt bespeaks the undue influence of a long tradition, in which the first correct solutions after almost a century of painful effort and fragmentary results (Forsyth 1890) have perhaps tended to dominate the scene even to the extent of discouraging the search for further improvement.

We shall now prove the required property of integrability on a manifold, extending it into time as well as space, and to anticlassical as well as classical systems (in which connexion the reader is referred back to § 3 for an explanation of the concepts of geometry as applied to anticlassical systems).

The property of nullity to be discussed is that provided by the generalized Hamilton equations (112). By eliminating the constrained displacements Δ from (112) we can express (112) through the linear system

$$\left.\begin{aligned} \mathsf{A}_{AB} X^B + (\lambda^e \Psi_e)_{,A} &= 0, \\ \Psi_{e,B} X^B &= 0, \end{aligned}\right\} \tag{118}$$

where the λ^e (anticommutative in the anticlassical case) are Lagrange multipliers, which are subject to no restriction beyond those encompassed in (118). Since $\Psi_e = 0$, there is no effective difference between $(\lambda^e \Psi_e)_{,A}$ and $\lambda^e \Psi_{e,A}$, and since the former is easier to manipulate, we use it.

By an argument given near the beginning of § 9, we may take the solutions X^A and λ^e of (118) to be differentiable fields, defined globally over the orbit manifold $\Psi_e = 0$.

Consider now two null fields $X_j{}^B$ and $X_k{}^B$, both satisfying (112) or (118). Our task will be accomplished if we can show that their Lie commutator $X_{[j,k]}{}^B$ (equation 84) also satisfies (112) or (118).

It is clear first of all that the Lie commutator of any two displacement fields in a manifold must

itself lie in the manifold. Therefore $X_{[j,k]}{}^{B}$ satisfies the second member of (118), as may also be verified by a trivial direct calculation.

To prove that it satisfies the first member of (118) we differentiate that equation in the manifold and multiply by two arbitrary displacements Δ_1 and Δ_2 in the manifold, to obtain

$$\{(BC)\ \mathsf{A}_{AB,C} X_j^B + \mathsf{A}_{AB} X_j^B{}_{,C} + (\lambda^e \Psi_e)_{,AC}\} \Delta_1^C \Delta_2^A = 0. \tag{119}$$

We then interchange Δ_1 and Δ_2, and subtract using (102), and apply the property of symmetry (95) and the property of closure (103) to what remains. From this calculation there emerges the intrinsic property (113), with however the field X of Lagrange indeterminacy in (113) now replaced by the general null field X_j.

We next take (113), replace Δ_1 by Δ and Δ_2 by X_k, interchange j and k, and subtract. Using the properties of symmetry and closure again, we find that

$$\begin{aligned} \Delta \times X_{[j,k]} &\equiv \mathsf{A}_{CB} X_{[j,k]}{}^{B} \Delta^C \\ &= -\mathsf{A}_{AB,C} \Delta^C X_j^B X_k^A - \mathsf{A}_{AB} X_j^B{}_{,C} \Delta^C X_k^A - \mathsf{A}_{AB} X_j^B X_k^A{}_{,C} \Delta^C \\ &\equiv -(\mathsf{A}_{AB} X_j^B X_k^A)_{,C} \Delta^C = 0. \end{aligned} \tag{120}$$

Thus the Lie commutator of two null fields is a null field.

Since the number of independent vectors X obeying (112) is necessarily finite in number, we may conclude that the null fields constitute a finite system closed under Lie commutation. From this, by Frobenius's theorem (Schouten & Kulk 1949*b*), it follows that the null vectors X of (112) integrate to form differentiable submanifolds, equal in dimension to the number of linearly independent X, and bounded only at points of singularity or degeneration of the Lagrange bracket.

We note in passing that equations (104) state that the primary displacements $X_{(1)}$ obey (118) with $\lambda = 0$. From this it now follows directly that every member of the Lie bracket sequence of the $X_{(1)}$ also obeys (118), and equivalently (112). This confirms the property of gauge invariance of the Lagrange bracket at fixed base point, previously established in (112) by a different argument. Thus both of the two properties of Lagrange gauge invariance (112) and (113) are independently confirmed, and moreover we now see that any differentiable solution X of (112) satisfies (113), whether it belongs to the Lagrange gauge set or not.

A fully constrained standard regular action principle for Dirac's extended motions

The space-time submanifolds which we have just found are the same as the motion submanifolds for any regular system (22), but for many singular systems, of which (115) is an example, they are larger. They accommodate Dirac's concept of gauge transformation and an associated concept of motion modulo the Dirac gauge group. Their differential characterization is provided by the equations (112) or (118). We shall now obtain a regular action principle for which (112) and the equations of constraint (98) are the Euler–Lagrange equations.

Let p_a, p_t and Ψ_e belong to any action principle (22), regular or singular, and consider in place of (22) the associated fully constrained action principle

$$\delta \int L_4 \, \mathrm{d}t = 0 \tag{121}$$

with

$$L_4 = p_a(t,q)\, \mathrm{D}q^a + p_t(t,q) - \lambda^e \Psi_e(t,q), \tag{122}$$

where, as is always necessary in the method of Lagrange multipliers, the Ψ_e have been manipulated to maximize the rank of $\Psi_{e,a}$.

The primary equations of (121) comprise

$$\mathsf{A}_{ab}\,\mathrm{D}q^b + \mathsf{A}_{at} + \lambda^e \Psi_{e,a} = 0, \tag{123}$$

$$\Psi_e = 0, \tag{124}$$

$$\Psi_{e,b}\,\mathrm{D}q^b + \Psi_{e,t} = 0. \tag{125}$$

By multiplying (123) by $\mathrm{D}q^a$ and (125) by λ^e and subtracting we can infer that the further equation

$$\mathsf{A}_{tb}\,\mathrm{D}q^b + \lambda^e \Psi_{e,t} = 0 \tag{126}$$

is automatically satisfied. Therefore, in view of (101), the primary equations of motion of (121) can be written in precisely the form (112) or (118), with however the evident proviso that we are demanding solutions X of (112) or (118) such that $X^t \neq 0$.

For a general bilinear form A the implications of the condition $X^t \neq 0$ would be far from trivial. Indeed, it is just this condition which is responsible for the generation of constraint chains (cf. § 5). However, in the present context we are assuming that the Ψ_e which appear in (122) have already been constructed from the action principle (22) by application of the method of causal analysis of § 2, and that they are a complete set, expressing fully the consequences of that analysis. It is therefore a property of the system (123)–(125) that it can be solved for all t by the orbits (25)–(26) of the action principle (22), at least one of which passes through every point of the manifold (124). The orbits of (22) obey (123)–(125) without imposition of any further constraints, and they obey them with $\lambda^e = 0$. So it follows that all of the constraints of the principle (121), other than those which fix some of the λ^e, are completely covered by (124). There are no secondary, etc., constraints limiting the q^a, and the only primary constraints to limit them are the set (124).

Let X^a denote a spatial solution of the system (118) or (112), i.e. a solution of (118) or (112) having $X^t = 0$. Then the most general possible solution $\mathrm{D}q^a$ of (123)–(125) can be obtained by taking a particular solution, for which any solution of (22) provides an obvious candidate, and modifying it by any such X according to the scheme

$$\mathrm{D}q^a \to \mathrm{D}q^a + X^a. \tag{127}$$

It is evident therefore that the orbits of the fully constrained action principle (121) embody the full Dirac gauge freedom of the bilinear form A_{ab} of (22) already at the level of primary indeterminacy. They do not produce any further gauge freedom in the motion of the q^a in higher orders of Lie commutation because, by the work of the previous subsection, all the relevant Lie commutators satisfy (118), and hence are already included among the X^a of (127).

Consider finally two neighbouring orbits of (121), between which an infinitesimal separation $\epsilon\eta^a$, $\epsilon\eta^e$ subsists. The η^a and η^e obey of necessity the Jacobi equations of (121). We must now examine these Jacobi equations, to see whether (121) is regular. It is evident by construction that the variation of η^e in the Jacobi action principle (cf. equation (16)) associated with (121) will produce correctly all the constraints (124) in the Jacobi form

$$\Psi_{e,a}\,\eta^a = 0. \tag{128}$$

Thus the Jacobi constraint system of (121) is not deficient where it concerns the η^a. At the same time, the Jacobi equations of (121) cannot possibly produce more constraints on the η^a than do the Lagrange equations of (121), because every orbit displacement of the latter system obeys

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them. Therefore the Jacobi equations of (121) produce correctly all the constraints relevant to the η^a, and produce no others which restrict the η^a in any way. This does not of course exhaust the set of constraints appropriate to the problem. For clearly (123)–(125) will in general lead to relations which fix some of the λ^e in terms of the q^a, and correspondingly we shall expect to find in the Jacobi equations certain constraints fixing some of the η^e in terms of the η^a. But it is not profitable for us to consider these just yet. Instead, we turn our attention to the Jacobi motion of the η^a and η^e. By the Jacobi version of the theorem of rank and nullity (§ 8) we know that this will be determinate to within the full latitude X allowed by the statement

$$\mathsf{A}_{ab} X^b \eta^a = 0 \quad (\forall \text{ Jacobi } \eta^a), \tag{129}$$

taken in conjunction with all the Jacobi constraints. It is to be noted that the η^e appear nowhere at all in (129) – the reason being that the kinetic part of the Jacobi action function does not depend upon them. Therefore, in considering the implications of (129), we can forget all about the η^e and the as yet unestablished status of their Jacobi constraint system. Only the η^a are relevant and for these we have just proved that the set of all Jacobi displacements $\epsilon\eta^a$ is the same as the set of all displacements Δq^a of the Lagrange orbit system of (121). From this it follows that the Lagrange bracket in (129) is identical with the Lagrange bracket (33) of (121). But we saw in (127) that the latter Lagrange bracket governs the indeterminacy of the Lagrange motion of (121). In short, the indeterminacy vectors X of the Jacobi system (129) span the same linear space as the Lagrange indeterminacy vectors X of (127). The Jacobi equations give an exhaustive description of the motion of the η^a.

Finally, we turn our attention to the η^e. These appear in the Jacobi equations only in the combination $\eta^e \Psi_{e,a}$. The linear independence of the $\Psi_{e,a}$ allows us always to express each η^e completely in terms of the η^a and the $\mathrm{D}\eta^a$. Since the Jacobi equations give a correct and exhaustive description of the motion and constraints of the latter variables we must now conclude that they also give a correct and exhaustive description of the motion and constraints of the η^e.

We have now proved that the Jacobi equations of (121) give a faithful and complete description of the kinematics and dynamics of all the orbit differentials η^a and η^e of (121). In other words, the completely constrained action principle (121) is regular, always.

By these considerations *any action principle* (1), *regular or singular, can be replaced by a standard fully constrained regular action principle* (121), *having the same Lagrange bracket and the same orbit manifold in* (t, q^a)*-space, and having in that space the same motion submanifolds as* (1) *if* (1) *is regular*. It is the latter fully constrained regular standard action principle, or one equivalent to it, which can be understood to lie behind Dirac's (1950, 1951, 1958, 1964*a*) theory of Poisson brackets, extended motion, and gauge.

The fully constrained action principle and (t, q^α)*-space*

There is in the constrained action principle (121) a great abundance of coordinates $q^{\alpha r}$, λ^e. Those orbits for which the λ^e are zero are orbits also of (22), and on them the equations of identification (17) or (20*a*) hold. These particular orbits are orbits of (1), and can be cast unambiguously into the smaller space (t, q^α) of the action principle (1). But when the solutions with $\lambda^e \neq 0$ are examined one finds that (17) invariably fail. Therefore it is impossible to describe the general orbit of (121) in (t, q^α) space. It has of course an image in that space, but the image is not complete, because it does not carry the variables $q^{\alpha r}$ on which the Lagrange bracket, etc., depend. To complete the image for an $M = 2$ system each orbit in (t, q^α)-space would have to be equipped at

each point with an arrow, tangential to the (t, q^α)-locus for orbits which are orbits of (1), but pointing otherwise for the remainder.

The failure of (17) in the Dirac extended motion of any indeterministic system with $M > 1$ is unavoidable. In any indeterministic system the primary indeterminacy appears at the level of the $\mathrm{D}^M q^\alpha$, and if $M > 1$ it reacts back, by integration of the equations of motion, to produce non-zero secondary indeterminacy in the $\mathrm{D}^{M-1} q^\alpha$, non-zero tertiary indeterminacy in the $\mathrm{D}^{M-2} q^\alpha$, etc. It is evident that in the larger (t, q^a)-space these various indeterminacies must arise at the various respective stages of Lie commutation, by which the hierarchy of indeterminacy vectors X is constructed. Now the concept of extended motion puts all these indeterminacies on the same footing, and makes them all simultaneously realizable by infinitesimal increment of t. Obviously this freedom of development is incompatible with the concept of motion as an evolving locus in (t, q^α)-space. For if $M > 1$, that concept embodies the idea that at any time t the variables $\mathrm{D}^{M-1} q^\alpha$, $\mathrm{D}^{M-2} q^\alpha$, ..., $\mathrm{D} q^\alpha$, q^α already have identifiable values, which can be ascertained by looking at the various curvatures of the locus. Under infinitesimal increment of t the only indeterminacy then permitted to appear is in the $\mathrm{D}^{M-1} q^\alpha$. For systems which are deterministic but singular and have $M > 1$ the same arguments and conclusions apply, except in those specially simple cases where the extended motion happens not to be indeterministic.

The Lagrange bracket in (t, q^α)-space

It will be appreciated that none of the above diminishes in any way the virtue of (112) as an invariant statement of a general concept of orbit displacement. Every displacement Δ and X of (112) can by derivation be interpreted as a displacement of orbit of (1) to orbit of (1). Thus, while (112) can be applied in (t, q^a)-space as the differential characterization of the extended motions of a motion submanifold, it can with equal facility be applied in both (t, q^a)- and (t, q^α)-space as the differential characterization of a set of integrable orbit to orbit displacements. And if the system (1) is regular, then all these orbit displacements can be realized by exploitation of its primary indeterminacy through finite periods of time.

Alternative formulations of the completely constrained action principle – non-canonical Hamiltonian formalism

The action principle (121) can be stated in terms of coordinates (t, x) intrinsic to the manifold (124). The Lagrange multiplier terms in (122) can then be discarded, without affecting either the orbits or the Lagrange bracket, and L_4 passes over into some function $L_5(t, x, \mathrm{D}x)$. If we arrange things suitably, we can take t, the x, and the values $\boldsymbol{\psi}_e$ of the functions Ψ_e, to constitute together a set of regular coordinates, at least for the orbit manifold and its first-order differential neighbourhood. And if we now add to L_5 any quadratic form $Q(\boldsymbol{\psi})$ in the variables $\boldsymbol{\psi}$, of maximal rank, then the resultant action principle

$$\delta \int \{L_5(t, x, \mathrm{D}x) + Q(\boldsymbol{\psi})\} \, \mathrm{d}t = 0 \tag{130}$$

will produce both the orbits and the constraints of (121). By expressing this action principle in terms of the coordinates q^a we can thus obtain a principle entirely equivalent to (121), yet containing no Lagrange multipliers. All of the constraints of this regular principle are primary.

Another more interesting possibility exists in which, while retaining certain Lagrange multipliers, we achieve a closer and less cumbersome contact with the variables of (1). Consider a constrained action principle (121) associated with (1) through (22), (19) and (18). The coordi-

nates q^a then comprise the $q^{\alpha r}$ $(r = 0, 1, \ldots, 2N-1)$ and the $\pi_{\alpha r}$ $(r = 0, 1, \ldots, 2N-2)$, which in (121) are supplemented by the further set of coordinates λ^e.

Among the constraints $\Psi_e = 0$ there now appear, by (21), a total of $\nu(2N-1)$ equations of the form

$$\pi_{\alpha r} - p_{\alpha r}(t, q^{\beta 0}, \ldots, q^{\beta\, 2N-1}) = 0 \quad (r = 0, 1, \ldots, 2N-2), \tag{131}$$

where, as previously noted, all $p_{\alpha r}$ with $r > N-1$ are identically zero. These constraints completely fix all the $\pi_{\alpha r}$, and therefore any other constraint equations of (22) can be expressed in the form

$$\psi_{e'}(t, q^{\alpha 0}, \ldots, q^{\alpha\, 2N-1}) = 0, \tag{132}$$

where e' ranges over a suitably reduced set of values. Clearly (132) are structurally the same as the equations of constraint of the system (1) in a treatment with $M = 2N$, and differ only in that the $q^{\alpha r}$ appear in place of the $\mathrm{D}^r q^\alpha$ as arguments.

In this system of coordinates the Lagrange function L_4 takes the form

$$L_4 = L_2 - \sum_{r=0}^{2N-2} \lambda^{\alpha r}\{\pi_{\alpha r} - p_{\alpha r}(t, q^{\beta 0}, \ldots, q^{\beta\, 2N-1})\} - \lambda^{e'}\psi_{e'}(t, q^{\beta 0}, \ldots, q^{\beta\, 2N-1}), \tag{133}$$

where L_2 is the same as in (19).

By prior variation of the $\lambda^{\alpha r}$ (or equally by prior variation of the $\pi_{\alpha r}$) the Lagrange function (133) can be seen to be entirely equivalent to the function

$$\begin{aligned} &L_6(t, q^{\beta 0}, \ldots, q^{\beta\, 2N-1}, \mathrm{D}q^{\beta 0}, \ldots, \mathrm{D}q^{\beta\, N-1}, \lambda^{e'}) \\ &\quad \equiv \sum_{r=0}^{N-1} p_{\alpha r}(t, q^{\beta 0}, \ldots, q^{\beta\, 2N-1})\, \mathrm{D}q^{\alpha r} + \left\{L_1(t, q^{\beta 0}, \ldots, q^{\beta N}) - \sum_{r=0}^{N-1} p_{\alpha r}(t, q^{\beta 0}, \ldots, q^{\beta\, 2N-1})\, q^{\alpha\, r+1}\right\} \\ &\qquad\qquad - \lambda^{e'}\psi_{e'}(t, q^{\beta 0}, \ldots, q^{\beta\, 2N-1}), \end{aligned} \tag{134}$$

both as regards its equations of motion and its Lagrange bracket. The latter, in its space-time version, is

$$\Delta_1 \times \Delta_2 = \sum_{r=0}^{N-1} \{\Delta_2 p_{\alpha r}\, \Delta_1 q^{\alpha r} - \Delta_1 p_{\alpha r}\, \Delta_2 q^{\alpha r}\} + \Delta_2 p_t\, \Delta_1 t - \Delta_1 p_t\, \Delta_2 t, \tag{135}$$

where p_t is the function inside the curly brackets in (134). This p_t is equal on orbit to the p_t of (22), thanks to the constraints (131).

For the systems of ordinary classical mechanics, where $N = 1$ and there are no $\psi_{e'}$, the regular action principle based on (134) embodies the same number of independent variables as appear in Hamilton's canonical equations (namely 2ν). It is therefore a worthwhile improvement upon (19), which contains also the redundant variables π. The equations of motion of (134) for these cases differ from Hamilton's canonical equations only by being written non-canonically, with the velocities $q^{\alpha 1}$ as basic variables in place of the Hamilton momenta $p_{\alpha 0}$.

To summarize this section, we now enunciate

THEOREM 6. *For the general system* (1), *whether regular or singular, the associated constrained first-order quasi-linear system* (134) *is regular, and its orbits are the orbits of Dirac's extended motion* (112), *which may best be calculated with the Lagrange bracket in the form* (135). *Those orbits which have* $\lambda^{e'} = 0$ *are the orbits of* (1). *If the system* (1) *is itself regular, then the extended motion* (112) *comprises no more than (and no less than) the set of all motion and gauge displacements of the associated unconstrained first-order quasi-linear action principle* (22). *And if* (1) *is both regular and deterministic, then all orbits of* (134) *have* $\lambda^{e'} = 0$, *and its set of orbits is the orbit set of* (1). *The Lagrange bracket of* (134) *is in all cases equal to that of* (1).

Various other choices of constrained action principle are also available. One can for instance treat (134) in the spirit of (130), and so remove the remaining Lagrange multipliers. Alternatively one can use intrinsic coordinates in the manifold $\psi_{e'} = 0$, and one can canonize these coordinates (§ 13).

One thing one cannot do, however, is to formulate a generally applicable concept of *completely* constrained action principle in the system of coordinates (t, q^α) of (1). Our concept of a partition of the Euler–Lagrange equations of (1) into two sets, to be counted respectively as equations of motion and equations of constraint, has no invariant connotation. It depends, for example, upon the choice of M. Moreover, no Lagrangian scheme formulated in (t, q^α) coordinates can ever produce the gauge motions which violate (17), and these motions of gauge seem to be involved in an essential way in the concept of completely constrained variational problem. And finally, a little experimentation with such simple Lagrange functions as

$$L_1 = q^2 \mathrm{D}q^1 + q^1 q^3 + (\mathrm{D}q^3)^2 \tag{136}$$

serves convincingly to show that modification of (1) by prior use of its constraints can damage both the Lagrange bracket and the Euler–Lagrange equations, even in cases which are both regular and deterministic. On the other hand, it is legitimate to vary (1) subject to its own *coordinate* constraints. This will be treated in a separate publication.

11. Some existence theorems for regular action principles

The property of complete closure

Every regular action principle (1) provides (i) a continuum of orbits and (ii) an invariant symplectic geometry on the continuum of orbits. In more detail, it gives

(i) a differentiable continuum of orbits in (t, q^α)-space, and a set of integrable displacements X of orbit initial value data, realizable by the motion and the indeterminacy (if any) of the orbits; equivalently an orbit manifold $\mathbb{M}$ embedded by the constraints into a topologically Euclidean space $\{t, q^{\alpha r};\ r = 0, 1, \ldots, M-1\}$, a differentiable continuum of orbits filling $\mathbb{M}$, and a stratification of $\mathbb{M}$ into motion submanifolds generated by the vectors X,

(ii) a set of differentiable and single-valued momentum functions $p_{\alpha r}$ (21) and p_t (135) in $\mathbb{M}$, and a derived antisymmetric bilinear differential form (135) having the properties (112) and (113) of invariance under motion and gauge and the property (88) of rank and nullity.

In the present section we shall show that the properties (i) and (ii) indicate always the existence of at least one regular action principle. However, our statement of property (ii) can be considerably sharpened and improved. For equation (135) makes it clear that one of the most salient features of the bilinear form of an action principle is that it is always a perfect differential, or curl. The alternative formula (101) makes no explicit reference to the p, and does not bring out this important property. Thus to sharpen the concept (ii) we must first ask what special features are required of a bilinear form (101), in order that it be possible to write it as a perfect differential or curl, as in (135). It is clear that the differential property of closure (103) will be necessary here, and it is indeed locally sufficient (Schouten & Kulk 1949*c*). However, it turns out that (103) is not by itself quite sufficient to ensure that (101) can be expressed globally by a formula of the type (135). For this to be possible an additional integral property of single-valuedness is required. We shall call the required integral property the property of *complete closure* (or *exactness*), and we shall formulate it in the next subsection.

The integral form of the Lagrange bracket

To facilitate the elucidation of the property of complete (or strong) closure or exactness we may introduce a single index ρ to range over the value t and the set of composite values αr. The bilinear form (135) then appears as

$$\Delta_1 \times \Delta_2 = \Delta_2 p_\rho(q) \,.\, \Delta_1 q^\rho - \Delta_1 p_\rho(q) \,.\, \Delta_2 q^\rho. \tag{137}$$

If we take three infinitesimally neighbouring points q^ρ, $q^\rho + \Delta_1 q^\rho$ and $q^\rho + \Delta_1 q^\rho + \Delta_2 q^\rho$ in $\mathbb{M}$, and use them in the order given as the successive vertices of an infinitesimal space-time triangle $\mathbb{T}$, then, by applying Simpson's rule to the sides of $\mathbb{T}$, we obtain

$$\oint_{\mathbb{T}} p_\rho \,\mathrm{d}q^\rho = \tfrac{1}{2}\Delta_2 \times \Delta_1. \tag{138}$$

This integral representation of the Lagrange bracket leads us to consider a finite shrinkable closed contour $\mathbb{C}$ in $\mathbb{M}$, which in the act of shrinking sweeps out some two dimensional region $\mathbb{D}$. By dividing $\mathbb{D}$ into small triangles, and using (138) to add their contributions, we obtain Stokes' theorem (Schouten & Kulk 1949*d*) in the form

$$\iint_{\mathbb{D}} \Delta_2 \times \Delta_1 = \oint_{\mathbb{C}} p_\rho \,\mathrm{d}q^\rho = \mathbf{P}[\mathbb{C}], \text{ say.} \tag{139}$$

In writing this formula the factor $\frac{1}{2}$ has been dropped in deference to the conventions commonly used in the theory of integration. Thus $\Delta_2 \times \Delta_1$ represents in (139) the contribution of an infinitesimal parallelogram, and $\iint$ means a sum over parallelograms. The functional notation $\mathbf{P}[\mathbb{C}]$ introduced in (139) will be used later.

It is interesting that there is nothing in our derivation of Stokes' theorem to limit its application to classical manifolds. The theorem applies equally to manifolds of mixed classical/anticlassical character. The same is true of the higher analogues of Stokes' theorem, which can be derived by suitably extending the above type of analysis. The point is that Simpson's rule with its obvious generalizations reduces these theorems of the integral calculus to algebraic statements, which depend only upon the rules of addition and multiplication, and not upon the precise nature of the symbols being added and multiplied (subject, of course, to existence of the appropriate limits).

According to equation (139) the double integral in its first member depends on the shape of the two dimensional region $\mathbb{D}$ only through its one dimensional boundary $\mathbb{C}$. This is the aforementioned property of complete closure – we shall show presently that any bilinear form (101) having this solenoidal property can be expressed as a perfect differential, as in (137).

The property of complete closure (139) is locally equivalent to the differential property of closure (103). Given two regions $\mathbb{D}$ and $\mathbb{D}'$ sharing the same boundary $\mathbb{C}$, one can try to prove the equality of the double integrals over $\mathbb{D}$ and $\mathbb{D}'$ by using a higher analogue of Stokes' theorem to convert the double integral over $\mathbb{D} - \mathbb{D}'$ into a triple integral over the region enclosed by $\mathbb{D} - \mathbb{D}'$. It is evident that this method of demonstration of (139) is effective whenever $\mathbb{D} - \mathbb{D}'$ is a shrinkable closed two-contour, i.e. whenever $\mathbb{D}$ and $\mathbb{D}'$ are homologous. But it is clear that nothing can be achieved in this way if there are boundary points of $\mathbb{M}$ inside $\mathbb{D} - \mathbb{D}'$. Thus the property of complete closure (139) is only equivalent at the local level to the differential condition (103), and at the global level it is generally more powerful than (103).

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The concept of single-valued continuous additive functional of a closed contour in a manifold

The functional $\mathbf{P}[\mathbb{C}]$ is Poincaré's circulation integral (Poincaré 1899; Cartan 1922; Abraham 1967), which in an approximate version of quantum theory is an integer multiple of Planck's quantum of action (Lanczos 1949). It is appropriate to call it the *Poincaré action*.

From a mathematical point of view the Poincaré action has the following salient properties:

(*a*) it is a single valued functional of the closed contour $\mathbb{C}$ (property of complete closure),

(*b*) it is additive under algebraic addition of contours,

$$\mathbf{P}[\mathbb{C}_1+\mathbb{C}_2] = \mathbf{P}[\mathbb{C}_1]+\mathbf{P}[\mathbb{C}_2], \tag{140}$$

(*c*) it is continuous; for a very small contour it scales quadratically under uniform dilation of the contour, and linearly under unidirectional dilation.

In our dynamical problem the functional $\mathbf{P}[\mathbb{C}]$ has also the property of motion and gauge invariance. Every contour $\mathbb{C}$ in $\mathbb{M}$ cuts through a succession of motion submanifolds. The value of $\mathbf{P}[\mathbb{C}]$ depends only upon the relevant sequence of motion submanifolds, and not upon the disposition of the points of intersection within their respective motion submanifolds. Finally, $\mathbf{P}[\mathbb{C}]$ has the property of rank. For any infinitesimal contour $\mathbb{T}$ in the plane of Δ_1 and Δ_2, one has

$$\{\mathbf{P}[\mathbb{T}] = 0;\ \forall\Delta_2\} \Rightarrow \{\Delta_1 = 0 \bmod X\}. \tag{141}$$

Thus every regular action principle (1) provides an orbit manifold, etc., as in (i) above, and also

(ii′) an additive single-valued continuous functional $\mathbf{P}[\mathbb{C}]$ in $\mathbb{M}$, having the properties of invariance and rank just described.

In the next subsection it will be proved that (ii) and (ii′) are interchangeable and fully equivalent, i.e. that (ii′) implies the existence of momentum functions p, etc. But before proceeding to this it will be useful to consider to what extent we are free, given (i), to make an assignment (ii′) to fit (i).

One necessary condition, which applies only in the classical case, is that the dimension of $\mathbb{M}$ should exceed the dimension of the motion submanifolds by an even integer (see equation (117)). Apart from this, it will be seen that the problem is one of assigning a $\mathbf{P}[\mathbb{C}]$ on a reduced manifold $\hat{\mathbb{M}}$ obtained by regarding each motion submanifold of $\mathbb{M}$ as a point. For deterministic systems $\hat{\mathbb{M}}$ can be taken to be simply a fixed-time section of $\mathbb{M}$. Essentially, therefore, the problem is one of assigning a Lagrange bracket at *one fixed time*, and subject to conditions of rank and single-valuedness. If the topology of $\hat{\mathbb{M}}$ is Euclidean this problem has huge numbers of solutions. For example, one may take any set of regular curvilinear coordinates in $\hat{\mathbb{M}}$ and lay out with respect to these a canonical (§ 12) assignment of $\mathbf{P}[\mathbb{C}]$. If $\hat{\mathbb{M}}$ is not Euclidean there may be certain topological conditions to be met, in default of which no assignment of $\mathbf{P}[\mathbb{C}]$ can be extended globally in a single-valued way. Such conditions, if any, are beyond the scope of this paper. But quite generally, we may surmise that if a system (i) admits a $\mathbf{P}[\mathbb{C}]$ at all, then it admits an infinity of different and inequivalent $\mathbf{P}[\mathbb{C}]$. This, as we shall see later, implies that if (i) can be derived from some regular action principle, then it can be derived from an infinity of different regular action principles having different Lagrange bracket systems.

Existence of the momenta – a theorem of de Rham

Let us suppose now that we are given an orbit system (i) already endowed with a single-valued additive continuous functional (ii′). Then in any finite topologically Euclidean local region

$\mathbb{M}'$ of $\mathbb{M}$ we can construct an extension $\mathbf{P}'[\mathbb{AB}]$ of $\mathbf{P}[\mathbb{C}]$ from the set of all closed contours $\mathbb{C}$ in $\mathbb{M}'$ to the set of all open curvilinear arcs $\mathbb{AB}$ in $\mathbb{M}'$, having the property of additivity and continuity with respect to algebraic addition of open arcs, and having also the property that for any closed arc $\mathbb{AA} \equiv \mathbb{C}$,

$$\mathbf{P}'[\mathbb{AA}] = \mathbf{P}[\mathbb{AA}]. \tag{142}$$

Such an extension can be achieved by taking a base point $\mathbb{O}$ in $\mathbb{M}'$, and drawing from it a differentiable continuum of rays to fill $\mathbb{M}'$ (the Poincaré star construction). If $\mathbb{OA}$ denotes a segment of ray running from $\mathbb{O}$ to $\mathbb{A}$, etc., then the formula

$$\mathbf{P}'[\mathbb{AB}] = \mathbf{P}[\mathbb{OA} + \mathbb{AB} + \mathbb{BO}] \tag{143}$$

furnishes a suitable $\mathbf{P}'$. It is obvious that it is additive and continuous, and if $\mathbb{B}$ approaches $\mathbb{A}$ it tends, by additivity of $\mathbf{P}$, to

$$\mathbf{P}'[\mathbb{AA}] = \mathbf{P}[\mathbb{OA} + \mathbb{AO}] + \mathbf{P}[\mathbb{AA}] = \mathbf{P}[\mathbb{AA}]. \tag{144}$$

Since any additive continuous functional $\mathbf{P}'$ defined on all open arcs $\mathbb{AB}$ admits of necessity a representation

$$\mathbf{P}'[\mathbb{AB}] = \int_{\mathbb{A}}^{\mathbb{B}} p'_\rho \, \mathrm{d}q^\rho, \tag{145}$$

the representation

$$\mathbf{P}[\mathbb{C}] = \oint_{\mathbb{C}} p'_\rho \, \mathrm{d}q^\rho \tag{146}$$

is hereby achieved over $\mathbb{M}'$. The momenta p'_ρ in (145) are of course unique to the $\mathbf{P}'$ considered, to within the transformation

$$p'_\rho \to p'_\rho + \Lambda^{e'} \psi_{e',\rho} \tag{147}$$

with arbitrary Λ, but those in (146) can be further modified according to the scheme

$$p'_\rho \to p'_\rho + \Lambda^{e'} \psi_{e',\rho} + \Theta_{,\rho}, \tag{148}$$

where Θ is any differentiable function over $\mathbb{M}'$.

Let p' and p'' denote possible momentum assignments in two finitely overlapping regions $\mathbb{M}'$ and $\mathbb{M}''$. Irrespective of the topology of the region of overlap, the property of single valuedness of $\mathbf{P}$ (property of complete closure) implies for any $\mathbb{C}$ in that region the equalities

$$\oint_{\mathbb{C}} p'_\rho \, \mathrm{d}q^\rho = \mathbf{P}[\mathbb{C}] = \oint_{\mathbb{C}} p''_\rho \, \mathrm{d}q^\rho. \tag{149}$$

Therefore there exists over the whole overlap region a single-valued function Θ, given by

$$\Theta(q) = \int^{q} \{p''_\rho(q') - p'_\rho(q')\} \, \mathrm{d}q'^\rho. \tag{150}$$

After extending this Θ as we please to the whole of $\mathbb{M}'$, we can exploit it in (148) to make p' and p'' the same throughout the whole region of overlap. In this way the region $\mathbb{M}' \cup \mathbb{M}''$ is now covered by a single p assignment. This procedure can be repeated until the whole of $\mathbb{M}$ is covered.

These considerations provide straightforward proof of a theorem of de Rham (1931, see also Hodge 1941), and extend it to anticlassical and mixed manifolds. The other higher versions of de Rham's theorem extend with equal facility. As with Stokes' theorem, the reason that de Rahm's theorems do extend is that they are essentially algebraic.

The theorem of Lie and Koenigs

Besides the global theorem of de Rham and the local star construction of Poincaré, there is also a closely related local theorem of Lie and Koenigs (see Whittaker 1937). After transcription from canonical to general coordinates that theorem states that to any set of deterministic unconstrained orbits endowed with a conserved fixed-time Lagrange bracket (33) there belongs locally a function p_t, such that the motion is subject to (112). This theorem *assumes* the existence of all the p_ρ other than p_t, and in that respect, as in several others, is obviously less powerful than the de Rham theorem.

In its original statement (see Whittaker 1937) the Lie–Koenigs theorem made much use of the concept of canonical coordinates. However, its canonical aspects are logically distinct from the present discussion. They are completely covered by the work of § 13.

Existence of a regular action principle

The results now established can best be interpreted by regarding the additive functional $\mathbf{P}[\mathbb{C}]$ as defined in the space $\hat{\mathbb{M}}$, wherein each motion submanifold of the orbit set (i) is represented as a single point. Once a $\mathbf{P}[\mathbb{C}]$ of maximal rank is assigned in $\hat{\mathbb{M}}$, then by invariance it is assigned on every $\mathbb{C}$ of $\mathbb{M}$. Therefore functions p_t and $p_{\alpha r}$ exist throughout $\mathbb{M}$, and they are unique to within the latitude allowed by (148). By construction they obey the property of rank and nullity (88), and the conditions of motion invariance (113). Conversely, if we now take these functions p, and use them in (112) to provide a system of differential equations for the orbit submanifolds, then we can be sure that at every point of $\mathbb{M}$ these equations have solutions X with $X^t \neq 0$, and that no further constraints arise. Consequently, just as with (121), we can now argue that (112) are the Euler–Lagrange equations of a quasi-linear first-order constrained action principle with

$$L_7 = \sum_{r=0}^{M-1} p_{\alpha r}(t, q^{\beta s})\, \mathrm{D}q^{\alpha r} + p_t(t, q^{\beta s}) - \lambda^{e'} \psi_{e'}(t, q^{\beta s}) \tag{151}$$

as Lagrange function. And we can argue that this action principle is regular, for precisely the same reasons that (121) is regular. Obviously, too, it yields the correct Lagrange bracket, and continues to yield it correctly under any modification (148) of the p. Since the p are unique to within (148), this represents the maximum possible freedom allowed to an L_7 of this type. Any other modification of the factors p within the quasi-linear first-order framework would upset the Lagrange bracket of L_7. It is noteworthy that the modification (148) changes L_7 only by addition of a perfect differential

$$L_7 \to L_7 + \mathrm{D}(\Lambda^{e'} \psi_{e'} + \Theta). \tag{152}$$

Conversely, the addition of a perfect differential effects always a transformation (148).

Although in any actual problem we would use the smallest compatible M, it is profitable to consider what happens if we increase M beyond this. Clearly more variables and a corresponding number of constraints come in. But the Lagrange bracket continues to be correctly represented if the new p are all set equal to zero and the old p are kept as before, in which case the value of L_7 in $\mathbb{M}$ is not changed. Therefore, by (152), the value of L_7 in $\mathbb{M}$ is unique to within a perfect differential, irrespective of the choice of M. Thus we have

THEOREM 7. *Both the orbits and the Lagrange bracket of any orbit system endowed with a motion and gauge invariant single-valued additive continuous functional* $\mathbf{P}[\mathbb{C}]$ *subject to the appropriate condition of*

maximal rank can be obtained, modulo the gauge group, from a regular quasi-linear first-order Lagrange function (151), *and the value of* (151) *in* $\mathbb{M}$ *is unique to within a perfect differential, irrespective of* M.

Action principles in (t, q^α)*-space – an embedding theorem*

The regular principle (151) bears witness to the universal scope of the principle of stationary action. Nevertheless it is clumsily contrived, and with its highly redundant set of variations $\delta q^{\alpha r}$, $\delta\lambda^{e'}$ it lacks the economy and brevity of principles like (1). And for indeterministic systems it has the further disadvantage that not all its orbits can be represented in the basic (t, q^α)-space of the given set of orbits and Lagrange brackets. One would much prefer a regular action principle of the general form (1), set in the coordinates most natural to the dynamical problem. However, we shall show by example that there are cases wherein the given coordinates q^α are too few for their variations to specify the given orbits and Lagrange brackets through any action principle (1). We shall prove that it is always possible to *embed* the given orbits and their Lagrange brackets into the orbits and Lagrange brackets of some regular system (1) in the given coordinates, but that in some cases this embedding is necessarily non-trivial.

Our discussion of this matter will be very tentative and will do no more than illustrate the extreme difficulty of the general problem. We start by appealing to a feature already tacitly established in §§ 2–10, namely that if the given (t, q^α)-orbits (i) and their Lagrange bracket (ii) can be obtained from a regular action principle (1) with Lagrange function $L_1(t, q^\alpha, \ldots, \mathrm{D}^N q^\alpha)$ for some N (necessarily $\leqslant M$), then their motion submanifolds and Lagrange bracket can be obtained from an associated regular quasi-linear first-order action principle (134) whose Lagrange function *on orbit* is equal in value to L_1. In view of the uniqueness of the value established through the de Rham theorem at the end of the previous subsection, we may therefore equate L_1 to (151) on orbit, to within a perfect differential. From this we infer that *the hypothetical function* L_1, *if indeed it exists at all, must be of the form*

$$\begin{aligned} L_1(t, q^\beta, \ldots, \mathrm{D}^N q^\beta) = {} & \sum_{r=0}^{M-1} p_{\alpha r}(t, q^\beta, \ldots, \mathrm{D}^{M-1} q^\beta)\, \mathrm{D}^{r+1} q^\alpha + p_t(t, q^\beta, \ldots, \mathrm{D}^{M-1} q^\beta) \\ & + \mathrm{D}\Theta + \mu^{e'}(t, q^\beta, \ldots, \mathrm{D}^M q^\beta)\, \psi_{e'}(t, q^\beta, \ldots, \mathrm{D}^{M-1} q^\beta) \\ & + \nu^f(t, q^\beta, \ldots, \mathrm{D}^M q^\beta)\, \omega_f(t, q^\beta, \ldots, \mathrm{D}^M q^\beta), \end{aligned} \tag{153}$$

where the functions p *are any suitable continuations of the functions which appear in* (151), $\mathrm{D}\Theta$ *is any perfect differential, the factors* $\mu^{e'}$ *and* ν^f *are functions yet to be determined, the* $\psi_{e'}$ *are the given constraint functions in an analysis with index* M, *and the* ω_f *are functions which, when equated to zero, express the remaining equations of motion of the given orbits* (i).

The function Θ can be chosen quite at will. A change of Θ affects neither the equations of motion nor the Lagrange bracket. All that it does is to alter the functions p, and these are subject to (148) anyway. So we shall choose Θ to be zero henceforth.

In analysing the implications of the proposed action principle (153) the space-time paths to be considered are of course in (t, q^α)-space, and along them $\mathrm{D}^r q^\alpha$ and $q^{\alpha r}$ are one and the same. To increase the visual impact of the formulae, and make clear the connexion with the theory of earlier sections, we shall therefore use the notation $q^{\alpha r}$ where it is advantageous to do so.

Let us consider a small spatial orbit to orbit displacement Δ. Since the functions ψ and ω are zero for the given orbits we read off from (153) that

$$\Delta L_1 = \sum_{r=0}^{M-1} (\Delta p_{\alpha r} . \mathrm{D} q^{\alpha r} + p_{\alpha r} \Delta q^{\alpha\, r+1}) + \Delta p_t. \tag{154}$$

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Now the given motion D satisfies (112), and therefore we also have

$$\sum_{r=0}^{M-1} (\Delta p_{\alpha r} . \mathrm{D}q^{\alpha r} - \mathrm{D}p_{\alpha r} . \Delta q^{\alpha r}) + \Delta p_t = 0. \tag{155}$$

By comparing these two equations we can infer that the function (153) satisfies the system

$$\Delta L_1 = \sum_{r=0}^{M-1} (\mathrm{D}p_{\alpha r} . \Delta q^{\alpha r} + p_{\alpha r} \Delta q^{\alpha\, r+1}) \tag{156}$$

for all allowed spatial Δ, at every point of every given orbit. Since the displacements Δ are inhibited only by the vanishing of the functions ψ and ω it follows from (156) that at every point of every given orbit there exists a unique choice of the multipliers $\mu^{e'}$ and ν^{f}, such that equations (20*b*) are obeyed, with the π of (20*b*) now replaced by the functions p which we have chosen to represent the given Poincaré action $\mathbf{P}[\mathbb{C}]$, and L_1 in (20*b*) standing now for the function (153).

But equations (20*b*) comprise both the Euler–Lagrange equations (6) of the action principle (1), and also the defining equations (21) of its momenta. *With the indicated choice of the functions μ and ν* (which are unique on orbit once the p have been assigned on $\mathbb{M}$ and extended into its first-order differential neighbourhood) *we have* therefore *an action principle in (t, q^α)-space, whose orbits include all the given orbits, and whose Lagrange bracket reduces to the given Lagrange bracket for all the given orbits.* This action principle may or may not be regular, depending on how we choose to continue the functions μ and ν to points off orbit. Even if it is regular it may possess orbits over and above those which are given, in which case the given system is embedded in the system of orbits and Lagrange brackets of (153). Sometimes there is no continuation of μ and ν which avoids this weakness of (153), and the only way to overcome it is to use more coordinates in the statement of the action principle.

Examples

As a first example we take an $M = 4$ classical constraint-free system with equation of motion

$$\mathrm{D}^4 q - q = 0 \tag{157}$$

and regular momentum assignment

$$p_0 = \mathrm{D}q, \quad p_1 = 0, \quad p_2 = \mathrm{D}^3 q, \quad p_3 = 0 \tag{158}$$

for all t. It is readily verified that the fixed-time Lagrange bracket is conserved. To find p_t it is sufficient, following Lie and Koenigs (see Whittaker 1937), to integrate the differential identity (155). This yields

$$p_t = -\tfrac{3}{2}(\mathrm{D}q)^2 + \mathrm{D}(q\,\mathrm{D}q) - \tfrac{1}{2}(\mathrm{D}^3 q)^2. \tag{159}$$

Therefore, by (153), the only possible Lagrange function in the variables (t, q) is given, apart from a perfect differential, by the formula

$$L_1 = -\tfrac{1}{2}(\mathrm{D}q)^2 + \tfrac{1}{2}(\mathrm{D}^3 q)^2 + \nu . (D^4 q - q). \tag{160}$$

This fits equations (20*b*) only if $\nu = 0$ on orbit. This means that ν must contain a factor $\mathrm{D}^4 q - q$. If ν is not chosen identically zero, the Euler–Lagrange equation will therefore contain a term in $\mathrm{D}^8 q$. This is not desired. The best choice is therefore

$$\nu \equiv 0. \tag{161}$$

The Euler–Lagrange equation is then

$$\mathrm{D}^6 q - \mathrm{D}^2 q = 0, \tag{162}$$

which provides us with a plain example of a non-trivial embedding. This is the best that can be done with one variation δq.

As a second example we take again a classical system with equation of motion (157), but this time with the regular momentum assignment

$$p_0 = \mathrm{D}^3 q, \quad p_1 = -\mathrm{D}^2 q, \quad p_2 = p_3 = 0. \tag{163}$$

The Lagrange bracket is again conserved, and (155) yields

$$p_t = \tfrac{1}{2}q^2 + \tfrac{1}{2}(\mathrm{D}^2 q)^2 - \mathrm{D}q\,\mathrm{D}^3 q. \tag{164}$$

Again equations (20b) necessitate that $\nu = 0$ on orbit, and again the best choice of ν is $\nu \equiv 0$. With this choice we get

$$L_1 = \tfrac{1}{2}q^2 - \tfrac{1}{2}(\mathrm{D}^2 q)^2, \tag{165}$$

which has (157) as its Euler–Lagrange equation, and so provides a proper description, without embedding.

From these two simple examples it will be seen that even in the absence of indeterminacy there are some problems of classification yet to be resolved. We do not have as yet any general criterion by which to reckon the minimum number of variations needed in an action principle, and so distinguish, for example, between (158) and (163).

Other types of embedding

We have so far discussed only the embedding of systems with regular Lagrange bracket assignment, i.e. an assignment obeying the full condition of rank (88). It is also possible to embed certain singular systems. Consider for example the deterministic classical orbit system $\mathrm{D}q = 0$. Since any antisymmetric matrix in one dimension vanishes, it is not possible to assign to it any Lagrange bracket meeting the requirement of rank. Therefore $\mathrm{D}q = 0$ cannot be derived from a regular action principle involving only the variable q and its derivatives. Nor can it be derived from any singular action principle involving only the variable q and its derivatives. But it can be embedded in the regular Lagrangian system $L_1 = \frac{1}{2}(\mathrm{D}q)^2$, and in many other regular systems also.

Regular embeddings of systems of deficient rank are of great value in the theory of electrodynamics. They were first introduced by Fermi (1929), who used them to give a manifestly relativistic formulation of the Poisson bracket structure. It is also possible to accommodate the deficient rank structure of electrodynamics without embedding, by relaxing the requirement of determinacy. This alternative approach has been formulated by Dirac (1952), through his concept of extended motions. Unfortunately the extended motions do not exhibit explicit Lorentz invariance. Yet another possibility is to discard all variables other than the gauge invariants. But here the difficulty is that there exists no complete set of gauge invariants with simple Lorentz transformation properties. Thus embedding is by far the most satisfactory possibility.

12. The solution of Pfaff's problem on classical and anticlassical manifolds by the method of coordinate lines

Pfaff's problem as a problem of a bilinear form

The general aim and purpose in Pfaff's problem is to simplify the linear differential form

$$p_A(q)\,\Delta q^A \tag{166}$$

by means of a regular coordinate transformation $q^A \to y^A$. Now with every such linear form there is a derived antisymmetric bilinear form (101), whose matrix A_{AB} (95) obeys the differential condition of closure (103) and the stronger condition of complete closure (single-valuedness of a double integral (139) with respect to all possible regions $\mathbb{D}$ of boundary $\mathbb{C}$). If we are given the A_{AB}, we can recover the p_A from them, to within a perfect differential, etc. (148), by applying (139) and the de Rham theorem (146). Thus the problem of simplifying the linear form (166) is almost completely resolved if we know how to simplify the associated bilinear form.

Amputation of the coordinates of motion and gauge

In our dynamical problem the linear form (166) and the derived bilinear form (101) are defined on a global manifold $\mathbb{M}$ of dimension $\mathscr{D}+\mathscr{I}+1$, whose first $\mathscr{D}+\mathscr{I}$ dimensions may be classical or anticlassical (cf. equation (116)). But in either case the theorem of integrability of the null vectors of a closed bilinear form on a manifold (equation (120)) tells us that $\mathbb{M}$ can be stratified into motion submanifolds of dimension $\mathscr{I}+1$, each of which can be identified by a total of $\mathscr{D}$ independent *intrinsic* coordinates $Q^R(q)$ $(R = 1, \ldots, \mathscr{D})$ in $\mathbb{M}$. The number $\mathscr{D}$ here is the rank of A_{AB} in the manifold $\mathbb{M}$, and in the classical case it is even. The $\mathscr{D}$ coordinates Q^R can be chosen to be integrals of the motion (cf. equation (107)), i.e. they can be chosen to be $\mathscr{D}$ independent solutions of the integrable system of differential equations

$$Q^R{}_{,A} X^A = 0 \quad (\forall X). \tag{167}$$

To complete the assignment of new coordinates we need also a total of $\mathscr{I}+1$ further *intrinsic* coordinates Q^S $(S = \mathscr{D}+1, \ldots)$ not satisfying (167). Then movement within a motion submanifold is characterized by fixed Q^R and varying Q^S:

$$\{\Delta_1 \times \Delta_2 = 0;\ \forall \Delta_2\} \Leftrightarrow \{\Delta_1 Q^R = 0;\ \forall R\}. \tag{168}$$

It is to be noted that by using only $\mathscr{D}+\mathscr{I}+1$ coordinates we have already relinquished the possibility of a globally single-valued coordinatization. The theory to be developed is local, except in the simple case where $\mathbb{M}$ is simply connected.

The next step is to refer the linear form to the new intrinsic coordinates Q^R and Q^S

$$p_A \Delta q^A = P_R \Delta Q^R + P_S \Delta Q^S, \tag{169}$$

where, since we work in a manifold, $$\mathscr{V}_{e,A} \Delta q^A = 0. \tag{170}$$

Consider the Poincaré circulation (138) around a small triangle $\mathbb{T}$, one of whose sides is a displacement at fixed Q^R. This side is an X, and the circulation vanishes. Therefore, referring the triangle to the intrinsic coordinates Q^R and Q^S, we find

$$A_{RS} = 0, \quad A_{SS'} = 0 \quad (\forall R, S, S'). \tag{171}$$

The bilinear form is thus reduced to

$$\Delta_1 \times \Delta_2 = A_{RR'} \Delta_2 Q^{R'} \Delta_1 Q^R, \tag{172}$$

which does not depend upon the differentials of the Q^S. Moreover, in the new coordinate system the X can without loss of generality be represented by unit vectors pointing along the directions of the coordinate lines S, and therefore the invariance condition (113) (itself a consequence of (112) and the differential property of closure) reduces to

$$A_{RR',S} = 0 \quad (\forall R, R', S). \tag{173}$$

Thus the bilinear form does not depend upon the Q^s in any way at all. So far as the bilinear form is concerned, the manifold $\mathbb{M}$ has been replaced by a reduced manifold $\hat{\mathbb{M}}$ of dimension $\mathscr{D}$, with respect to which the form has rank $\mathscr{D}$. We must now apply the de Rham theorem (146), to recover the linear form (166) of which the Q^s-independent bilinear form (172) represents the circulation. Applying (146) first in $\hat{\mathbb{M}}$, we infer the existence of functions $\Pi_R(Q^{R'})$, such that

$$\mathsf{A}_{RR'} = \Pi_{R,R'} - (R'R)\,\Pi_{R',R}. \tag{174}$$

By applying the same theorem in the manifold $\mathbb{M}$, and appealing to the uniqueness property (148), we see that there must be some function $\Theta(Q^R, Q^s)$ such that

$$P_R = \Pi_R + \Theta_{,R}, \quad P_s = \Theta_{,s}. \tag{175}$$

If Θ happens not to depend on the Q^s then the terms $\Theta_{,R}$ can be absorbed into the Π_R, and the linear form (166) reduces through (169) to

$$\Pi_R(Q^{R'})\,\Delta Q^R, \tag{176}$$

which refers only to the intrinsic variables of $\hat{\mathbb{M}}$.

If Θ does depend on the Q^s the linear form cannot be reduced beyond the point

$$\Pi_R(Q^{R'})\,\Delta Q^R + \Delta\Theta(Q^R, Q^s). \tag{177}$$

In conventional Pfaff theory the function Θ in the latter case would be taken as a new coordinate, along with the Q^R and a suitable remaining set of Q^s.

The Jacobi identity

The matrix $\mathsf{A}_{RR'}$ has rank and dimension equal, and this enables us to introduce its transposed inverse $\mathsf{A}^{RR'}$, according to the scheme

$$\mathsf{A}^{R\bar{R}}\,\mathsf{A}_{R'\bar{R}} = \delta^R{}_{R'} = \mathsf{A}^{\bar{R}R}\,\mathsf{A}_{\bar{R}R'}. \tag{178}$$

By differentiating once we can express the derivative of the inverse in terms of the derivative of the original

$$\mathsf{A}^{R\bar{R}}\,\mathsf{A}^{R'R''}{}_{,\bar{R}} = -\,\mathsf{A}^{R\bar{R}}\,\mathsf{A}^{R'\bar{R}'}\,\mathsf{A}^{R''\bar{R}''}\,\mathsf{A}_{\bar{R}'\bar{R}'',\bar{R}}. \tag{179}$$

When we use this to write the closure property (103) in terms of the inverse, we obtain (since now all the indices are of one type, either classical or anticlassical)

$$\mathsf{A}^{R\bar{R}}\,\mathsf{A}^{R'R''}{}_{,\bar{R}} + \text{cyclic permutations} = 0. \tag{180}$$

This may properly be called the Jacobi identity (see Allcock 1973, 1975).

Canonization of the Q^R in the classical case - the method of coordinate lines

All the above analysis applies equally to classical and to anticlassical systems. The language is geometrically suggestive, but its actual content is algebraic; therefore the theory is valid in any context where the rules of algebra hold, subject of course to suitable assumptions of differentiability. But we now come to a definite parting of the ways. We wish to find a local coordinate transformation $Q^R \to y^R(Q)$ such that the Lagrange bracket assumes the canonical form

$$\Delta_1 \times \Delta_2 = \mathsf{C}_{RR'}\,\Delta_2 y^{R'}\Delta_1 y^R, \tag{181}$$

where C is the constant symmetric imaginary matrix (39) if the system is anticlassical, but the constant antisymmetric real matrix (40) if it is classical.

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In the classical case the coordinate lines of y^2 provide a direction field d_2 which, referred to the new coordinates y, takes the form

$$d_2 y^R = \epsilon \delta^{R2} \equiv \epsilon C^{RR'} \partial y^1 / \partial y^{R'}, \tag{182}$$

where $C^{RR'}$ is the transposed inverse of $C_{RR'}$ (cf. (178)) and ϵ is an infinitesimal equal to $d_2 y^2$. Because it is in tensor form, the equation (182) translates into the Q coordinates as

$$d_2 Q^R = \epsilon A^{RR'} \partial y^1 / \partial Q^{R'}. \tag{183}$$

Once we know the function $y^1(Q)$, this differential equation enables us to plot out the coordinate lines of y^2, even though we know nothing about $y^3, y^4, \ldots$. Moreover, it tells us that y^2 must increase uniformly along these lines, at a rate governed by

$$(\partial y^2 / \partial Q^R) A^{RR'} \partial y^1 / \partial Q^{R'} = 1. \tag{184}$$

Taking advantage of these features, we choose for y^1 any function we like (subject to $\partial y^1 / \partial Q^R \not\equiv 0$), fill $\hat{\mathbb{M}}$ with one dimensional integral manifolds of (183), mark off in some differentiable way a zero point for y^2 on each of them, and assign y^2 along them by (184).

With the differentiable function $y^2(Q)$ so manufactured, we can now construct a displacement field d_1, given by

$$d_1 Q^R = -\epsilon A^{RR'} \partial y^2 / \partial Q^{R'}, \tag{185}$$

and so determine also the coordinate lines of y^1. No conflict arises with our original assignment of the function $y^1(Q)$, because (184) automatically ensures that the y^1 we started with increases at precisely the correct rate along the coordinate lines (185). The crucial point, however, is whether the two sets of coordinate lines enmesh properly, as coordinate lines should. To study this we must look at the Lie commutator

$$d_{[1,2]} Q^R \equiv (A^{RR'} y^2{}_{,R'})_{,\bar{R}} A^{\bar{R}R''} y^1{}_{,R''} - (1 \rightleftarrows 2). \tag{186}$$

Fortunately it vanishes, by virtue of the Jacobi identity (180), the antisymmetry of A, and the first differential of (184). The proposed coordinate lines do indeed enmesh, and they mark out two dimensional surfaces in $\hat{\mathbb{M}}$, on which y^1 and y^2 are regular coordinates.

As a temporary mental aid we now introduce some $(\mathscr{D}-2)$ dimensional differentiable submanifold of $\hat{\mathbb{M}}$, coordinatized by a set of intrinsic variables $y^3, y^4, \ldots$, and such that each of the above two-manifolds intersects it at one point only. We continue $y^3, y^4, \ldots$ as constants over the various two-manifolds. Then $y^1, y^2, y^3, y^4, \ldots$ are regular coordinates, and $y^3, y^4, \ldots$ are independent solutions of the integrable system

$$\partial y^j / \partial Q^R A^{RR'} \partial y^k / \partial Q^{R'} = 0 \quad (k = 1, 2; j = 3, 4, \ldots). \tag{187}$$

These equations are in tensor form, and their left hand sides are form-invariant scalars under the change of coordinates $Q \to y$. Referred to the coordinates y they read, with (184),

$$A'^{21} = 1, \quad A'^{j1} = A'^{j2} = 0 \quad (j = 3, 4, \ldots), \tag{188}$$

where A' is the tensor transform of A.

Moreover, the $1jk$ and $2jk$ elements of the tensor equation (180) now become

$$A'^{jk}{}_{,1} = A'^{jk}{}_{,2} = 0 \quad (j, k = 3, 4, \ldots). \tag{189}$$

Thus coordinates 1 and 2 decouple from the rest, and it remains only to canonize the $\mathscr{D}-2$ remaining y.

It will be noted that this procedure gives an extremely effective construction, in that the successive equations for the final coordinates are all in tensor form, and can all therefore be referred directly to the coordinates Q, without the aid of the intermediary sets of coordinates invoked above. It is to be noted also that the bulk of the theory, from equation (178) onward, is totally insensitive to the presence of a perfect differential (175).

Comparison with other methods

The above treatment of the Pfaff problem compares favourably with other known methods, both as regards brevity and directness and as regards its constructive aspects. It is certainly less devious and more flexible than the Darboux method described by Goursat (1922*a*), and could well be said to be considerably easier than Cartan's Grassmann-algebraic method of divisors (Goursat 1922*a*; Thomas 1937; Cartan 1946; Schouten & Kulk 1949*a*; Slebodzinski 1963). One reason for this is that it makes full use of the appropriate tools of the calculus; tensors, Stokes' theorem, Lie brackets and de Rham's theorem. Another reason is that the gauge coordinates, which have no real bearing on the Pfaff problem, are cleared away at the beginning instead of at the end. Yet another point is that most of the constructions are insensitive to the addition of a perfect differential. In the last two features it differs also from Carathéodory's (1965) elegant function-theoretic method, though in truth Carathéodory's treatment is so remarkably different that a critical comparison seems impossible.

The method of coordinate lines admits a ready extension to anticlassical systems. In this it is distinctly better than the popular Cartan method. For the latter is based on the connexion between antisymmetric matrices A of even dimension and Pfaffians, and this connexion does not seem to have any analogue at all for symmetric A.

Canonization of the Q^R in the anticlassical case

For the anticlassical case we use arguments very similar in spirit to the foregoing. A displacement d_1 along a coordinate line of y^1, when referred to the y as coordinates, has the form

$$d_1 y^R = \epsilon \Upsilon \delta^{R1} = \pm \mathrm{i}\epsilon \Upsilon C^{RR'} \partial y^1/\partial y^{R'}, \tag{190}$$

where $\epsilon\Upsilon$ is an anticlassical infinitesimal equal to $d_1 y^1$, and the alternative signs allow for the two possible signs of a diagonal element of a canonical symmetric C (cf. equation (39)). The concept of 'coordinate line' is of course to be understood in an algebraic manner, as explained in § 3. Its import here lies in the differential equation (190) which, because it is in tensor form, can be expressed in terms of the Q as

$$d_1 Q^R = \pm \mathrm{i}\epsilon \Upsilon A^{RR'} \partial y^1/\partial Q^{R'}. \tag{191}$$

If the function $y^1(Q)$ were known, this equation would tell us how the Q^R change under change of y^1, holding the other y fixed. It would give us this information without our knowing what the other functions $y(Q)$ look like. And $y^1(Q)$ itself would change at a rate governed by

$$\pm \mathrm{i}\, \partial y^1/\partial Q^R A^{RR'} \partial y^1/\partial Q^{R'} = 1. \tag{192}$$

Thus the only permissible y^1 are solutions of the first-order partial differential equation (192).

We now discuss the solution of (192). The standard methods (Carathéodory 1965) for such equations carry over to anticlassical systems with only the most minor modifications. The procedure can be clarified by writing

$$\partial y^1/\partial Q^R = Y_R, \quad \text{say.} \tag{193}$$

Equation (192) can be regarded as providing at each point just one condition on the Y_R. However, since Y_R is a gradient, its value cannot be assigned from point to point at will. The condition (11) of anticommutativity of anticlassical differentiation limits the possibilities. To explore this, we differentiate (192) once, and use (11), and so obtain, with the help of the symmetry of A, the necessary condition

$$2Y_{R'',R}\,\mathsf{A}^{RR'}Y_{R'} = Y_R\,\mathsf{A}^{RR'}{}_{,R''}\,Y_{R'}. \tag{194}$$

This equation fixes the derivative of the gradient of y^1 along the characteristic direction $\Upsilon\mathsf{A}^{RR'}Y_{R'}$. In the same way (192), which can also be written as

$$\pm \mathrm{i}y^1{}_{,R}\,\mathsf{A}^{RR'}Y_{R'} = 1, \tag{195}$$

serves to fix the derivative of y^1 itself along the same characteristic direction. If therefore we assign y^1 at a point of $\hat{\mathbb{M}}$, and with it a vector Y_R obeying

$$\pm \mathrm{i}Y_R\,\mathsf{A}^{RR'}Y_{R'} = 1, \tag{196}$$

with the choice of sign here to be compatible with the signature properties of A, then (195) and (194) will propagate y^1 and Y_R away from that point along the appropriate characteristic line. Under this propagation the relation (196) is automatically preserved, because a displacement of magnitude

$$\Delta Q^R = \mathrm{i}\epsilon\Upsilon\mathsf{A}^{RR'}Y_{R'} \tag{197}$$

with Υ odd, induces in y^1 a change

$$\Delta y^1 = \pm\epsilon\Upsilon \tag{198}$$

and in $Y_{R''}$ a change

$$\Delta Y_{R''} = -\tfrac{1}{2}\mathrm{i}\epsilon\,\Upsilon Y_R\,\mathsf{A}^{RR'}{}_{,R''}\,Y_{R'} \tag{199}$$

since $Y_{R'',R}$ is also odd, and in $\mathsf{A}^{RR'}$ a change

$$\Delta\,\mathsf{A}^{RR'} = \mathrm{i}\epsilon\Upsilon\mathsf{A}^{RR'}{}_{,R''}\,\mathsf{A}^{R''R'''}Y_{R'''}, \tag{200}$$

and hence in $Y_R\,\mathsf{A}^{RR'}Y_{R'}$ no change at all. The picture is therefore one of a line, along which is strung a sequence of values of y^1, and with them a compatible set of assignments of $\partial y^1/\partial Q^R$; in short an anticlassical string of perforated coins, in which the plane of each coin represents a bit of surface of constant y^1, and the ordinal number of the coin represents y^1. Let us now take an analytic surface of dimension $\mathscr{D}-1$, and assign y^1 over it in an analytic manner. Then $\mathscr{D}-1$ components of $\partial y^1/\partial Q^R$ are fixed in the surface, and the one remaining component is fixed by (192). Thus Y_R is fixed by (193) on the surface. We now propagate a characteristic line from every point of the surface, and so attribute a value of y^1 and a value of Y_R to every point of a finite region of $\hat{\mathbb{M}}$. The y and Y obey (195). Therefore y will be a solution of (192) if it obeys everywhere the condition (193). At the beginning of each characteristic (193) is obeyed by construction. Also its component along the characteristic is obeyed everywhere, by virtue of the fact that (195) and (196) are both obeyed all the way. To investigate the remaining $\mathscr{D}-1$ components of (193), we consider two neighbouring characteristics, corresponding points of which are separated by a displacement δ. We wish to verify that

$$Y_R\,\delta Q^R - \delta y^1 = 0 \tag{201}$$

all the way, knowing that it is true at the beginning. Therefore, moving along the characteristic by the displacement Δ of (197)–(200), we must study $\Delta(Y_R\,\delta Q^R - \delta y^1)$ which, since $\Delta(\delta y) \equiv \delta(\Delta y)$, etc., may be written as $\Delta Y_R.\,\delta Q^R + Y_R\,\delta(\Delta Q^R) - \delta(\Delta y^1)$. On substituting (199) for ΔY_R, (197) for ΔQ^R, and (198) for Δy^1 we find that this does vanish, as required, so that (193) is indeed true sll

the way. Thus the construction provides everywhere a function y^1, obeying (193) and (195) and thence the partial differential equation (192). The coins on the various strings interleave properly to form integral surfaces, and the displacements which they represent are closed under Lie commutation.

Suppose therefore that we have now constructed a solution of the partial differential equation (192). We introduce a $(\mathscr{D}-1)$ dimensional analytic submanifold of $\hat{\mathbb{M}}$, endowed with intrinsic coordinates $y^2, y^3, \ldots$, and we draw out from each point of this surface a y^1 coordinate line (191). We take $y^2, y^3, \ldots$ to be constant along the lines (191). The manifold is thus coordinatized by a regular set of coordinates obeying

$$\pm \mathrm{i}\, \partial y^j/\partial Q^R\, \mathsf{A}^{RR'}\, \partial y^1/\partial Q^{R'} = \delta^{j1}. \tag{202}$$

Since this equation is in tensor form it transforms to

$$\pm \mathrm{i}\, \mathsf{A}'^{j1} = \delta^{j1}, \tag{203}$$

where A' is the representative of the tensor A with respect to the coordinate system y. Also, the $1jk$ element of the Jacobi identity (180) yields in the new coordinates the information

$$\mathsf{A}'^{jk}{}_{,1} = 0 \quad (j, k = 2, 3, \ldots). \tag{204}$$

Thus the coordinate y^1 is completely decoupled, and it remains only to canonize $y^2, y^3, \ldots$.

As in the classical case, all the differential equations for the final coordinates are in tensor form, and can be referred directly to the Q, without the aid of the intermediary coordinates invoked above. There is no unwanted material to throw away at the end of the canonization.

The linear form in canonical coordinates

In both cases the bilinear form is reduced finally to (181). To find the associated linear form we use (174) and (175) in the coordinates y. A solution to (174) is

$$\Pi_R = \tfrac{1}{2}\mathsf{C}_{RR'} y^{R'}. \tag{205}$$

So we have

$$p_A(q)\, \Delta q^A = \tfrac{1}{2}\mathsf{C}_{RR'} y^{R'} \Delta y^R + \Delta\Theta, \tag{206}$$

where Θ may or may not depend on coordinates other than the y^R (cf. (175) *et seq.*). If it does depend only on the y^R it can be removed by a suitable canonical transformation. The proof that this is possible for arbitrary $\Theta(y^R)$ in the classical case may be found in Carathéodory's (1965) book, and a proof for the anticlassical problem may be formulated in a similar way.

13. Hamilton's canonical equations for general variational principles

A time-dependent choice of the Q^R

In the above space-time canonization programme each motion submanifold of $\mathbb{M}$ is represented by a single point Q^R in the reduced manifold $\hat{\mathbb{M}}$. This reduction of dimension gives full expression to the various properties of conservation of the Lagrange bracket and the associated symplectic geometry of the orbit space. The variables Q^R of $\hat{\mathbb{M}}$ must properly be conceived as integrals (107) of the motion. It was this that led to the decoupling (171). Fortunately, however, the fact that we have appealed to the existence of $\mathscr{D}$ integrals of the motion does not imply that an explicit knowledge of $\mathscr{D}$ integrals is a necessary prerequisite for application of the canonization theory. Any set of $\mathscr{D}$ independent gauge invariant functions (106), taken at some chosen and

fixed instant of time, will serve to pick out a Dirac motion submanifold at that time and hence, implicitly, for all time. Thus it is quite sufficient to take as our integrals $Q^R(t, q^a)$ those particular integrals which, at the special instant $t = t_1$ of interest to us, coincide respectively with the members $G^R(t_1, q^a)$ of some given complete set of gauge invariants $G^R(t, q^a)$. Alternatively, we can say that we are looking at the canonization problem on a fixed-t section.

To each time t we therefore introduce a set of functions $G^R(t, q^a)$, continuous and differentiable in t, to serve modulo the gauge group in place of the q^a themselves at that t. And, because of the above considerations, we can at each instant t regard the $G^R(t, q^a)$ as a set of Q^R to be canonized. Thus we shall have many canonizations, one for each t and all on the same footing, and not just one as heretofore.

Formulation of the dynamics in terms of a complete set of gauge invariants

Consider a description of the motion in which $\mathscr{D}$ gauge-invariant functions $G^R(t, q^a)$ and $\mathscr{I}$ other functions serve in place of the q^a. Since the extended Lagrange bracket (101) obeys the conditions (112) of gauge invariance, it is possible to eliminate from it all differentials other than the ΔG^R and Δt, and so write

$$\Delta_1 \times \Delta_2 = \boldsymbol{A}_{RR'} \Delta_2 G^{R'} \Delta_1 G^R + \boldsymbol{A}_{Rt} \Delta_2 t \Delta_1 G^R + \boldsymbol{A}_{tR} \Delta_2 G^R \Delta_1 t. \tag{207}$$

The terms in t here arise because the G are not integrals of the motion; if they were integrals (207) would reduce to (172). The gauge invariance condition (113) tells us that the coefficients $\boldsymbol{A}$ depend only on t and the G^R, and not on any of the gauge coordinates of $\mathbb{M}$.

By de Rham's theorem functions $\boldsymbol{\Pi}_R(t, G)$ and $\boldsymbol{\Pi}_t(t, G)$ can be found, in terms of which (207) can be written as

$$\Delta_1 \times \Delta_2 = \Delta_2 \boldsymbol{\Pi}_R \Delta_1 G^R + \Delta_2 \boldsymbol{\Pi}_t \Delta_1 t - (1 \rightleftarrows 2). \tag{208}$$

The connexion between these and the p_A of (166) is that

$$p_a(t, q)\, \Delta q^a + p_t(t, q)\, \Delta t = \boldsymbol{\Pi}_R(t, G)\, \Delta G^R + \boldsymbol{\Pi}_t(t, G)\, \Delta t + \Delta\theta(t, q). \tag{209}$$

The completely constrained action principle (121) becomes

$$\delta \int \{\boldsymbol{\Pi}_R(t, G)\, \mathrm{D}G^R + \boldsymbol{\Pi}_t(t, G)\}\, \mathrm{d}t = 0, \tag{210}$$

after omission of the irrelevant perfect differential $\mathrm{D}\theta$.

Gauge invariance of p_t for time-independent systems

If the Lagrange function L_1 of the action principle (1) does not depend on t explicitly then the constraints will not depend on t explicitly, and the functions $G^R(t, q^a)$ can then be chosen to have no explicit t-dependence, and likewise the $\boldsymbol{\Pi}$. By the uniqueness property (209) it then follows that $\boldsymbol{\Pi}_t$ and p_t can differ at most by a constant, where p_t is defined directly in terms of the Lagrange function L_1 as explained under (135). Since $\boldsymbol{\Pi}_t$ is always gauge invariant, we have therefore the theorem that for time independent systems p_t is also gauge invariant. For time independent *regular* systems the same can more easily be deduced from the well known property that p_t is conserved in the absence of explicit time dependence.

To demonstrate that for time-dependent systems p_t is not necessarily gauge invariant we may consider the simple example

$$L_1 = \mathrm{D}\{q^2 f(t)\}, \tag{211}$$

where $f(t)$ is a varying function of t.

Canonization at fixed t: Hamilton's canonical equations

The equation of canonization (206) tells us that at each instant t we can introduce locally a set of canonical coordinates $y^R(t, G)$ appropriate to that particular t, and such that for any displacement Δ at fixed t we have

$$\boldsymbol{\Pi}_R(t, G)\,\Delta G^R = \tfrac{1}{2}C_{RR'}y^{R'}\Delta y^R. \tag{212}$$

For displacements Δ not at fixed t it is necessary to bear carefully in mind that the transformation $G \to y$ may depend explicitly on t. In this case the ΔG obeying (212) are not the actual displacements, but the displacements which would be produced by the same Δy at fixed t. Thus for displacements not at fixed t the implication is that

$$\boldsymbol{\Pi}_R(t, G)\,\{\Delta G_R - G^R(t, y)_{,t}\Delta t\} = \tfrac{1}{2}C_{RR'}y^{R'}\Delta y^R. \tag{213}$$

The action principle (210) reduces therefore to Hamilton's canonical form:

$$\delta\int\{\tfrac{1}{2}C_{RR'}y^{R'}\mathrm{D}y^R - H(t, y)\}\,\mathrm{d}t = 0, \tag{214}$$

where the Hamilton function H is given by

$$-H(t, y) = \boldsymbol{\Pi}_t(t, G) + \boldsymbol{\Pi}_R(t, G)\,\partial G^R(t, y)/\partial t, \tag{215}$$

and where, by (209), $\boldsymbol{\Pi}_t(t, G)$ is given by

$$\boldsymbol{\Pi}_t(t, G) = p_t - \frac{\partial\theta(t, q)}{\partial t} + \left\{p_a - \frac{\partial\theta(t, q)}{\partial q^a}\right\}\frac{\partial q^a(t, G, Q^S)}{\partial t}. \tag{216}$$

For time independent systems all the relevant transformations can be chosen to be time independent, in which case H becomes equal in value to $-p_t$. Now in the time independent case (and only in that case) the function p_t has an invariant and intrinsic significance, as the generator of a symmetry operation of the orbit system. For time dependent systems it is usually impossible to arrange equality between H and $-p_t$. Indeed, it is easy to see, by considering a displacement Δ_1 at fixed y and a displacement Δ_2 at fixed t, that such equality would in most cases be incompatible with the rank property of the Lagrange bracket. It is evident also from (215) and (216) that for time-dependent systems H depends perforce upon arbitrary details of time dependence in the transformations $q \to G \to y$. H has in general no invariant significance, and, in view of the ubiquitous perfect differential (as in (211)), the same can even be said of p_t.

A highly non-trivial worked example of the canonization of a constrained Lagrangian system may be found in a paper by Gupta & Repko (1969). Their findings with regard to H and p_t fully confirm the general remarks made above.

Symmetrical treatment of t and the gauge coordinates

A canonization procedure more flexible and less demanding than the foregoing may be obtained by introducing $\mathscr{I}+1$ motion variables $\tau^S(q^A)$, one of which may be t itself, to serve symmetrically as coordinates within the motion submanifolds, and a further set of $\mathscr{D}$ functions $F^R(q^A)$, not necessarily gauge invariant, to complete the coordinatization of $\mathbb{M}$. Then any motion submanifold can be located by giving a set of values for the τ^S, and with them the values f^R which the $F^R(q^A)$ assume at a point with coordinates τ^S on the submanifold in question.

The f^R for some chosen and fixed τ^S thus characterize and fix a whole motion submanifold,

and may therefore be regarded as a suitable set of integrals Q^R of the motion, ripe for canonization. Alternatively, we can argue that we are looking at the canonization problem on a fixed-τ section. The Lagrange bracket can be expressed in terms of the f^R and the τ^S, much as in (207), with coefficients $\boldsymbol{A}$ depending on the f and the τ. Equation (208) passes over to an analogous equation of the form

$$\Delta_1 \times \Delta_2 = \Delta_2 \boldsymbol{\pi}_R(\tau, f) \,.\, \Delta_1 f^R + \Delta_2 \boldsymbol{\pi}_S(\tau, f) \,.\, \Delta_1 \tau^S - (1 \rightleftarrows 2), \tag{217}$$

and the completely constrained action principle (121) for Dirac's extended motion (112) becomes

$$\delta \int \{\boldsymbol{\pi}_R(\tau, f)\, \mathrm{D}f^R + \boldsymbol{\pi}_S(\tau, f)\, \mathrm{D}\tau^S\}\, \mathrm{d}t = 0, \tag{218}$$

after omission of a perfect differential. Formally one would here vary both the f^R and $\mathscr{I}$ of the τ^S. However, it is clear from the gauge and motion invariance of the Lagrange bracket that the variation of the τ^S provides no information extra to that provided by variation of the f^R. By the same token we see at once that the indeterminacy in the solution of the Euler–Lagrange equations (112) of (218) consists precisely in a freedom to assign at will the magnitudes of $\mathscr{I}$ of the $\mathrm{d}\tau^S \equiv (\mathrm{D}\tau^S)\, \mathrm{d}t$.

For every fixed set of values of the τ we can now introduce a new set of canonical coordinates $y^R(\tau, f)$ peculiar to that set of values and such that, for any displacement Δ at fixed τ, we have

$$\boldsymbol{\pi}_R(\tau, f)\, \Delta f^R = \tfrac{1}{2} \boldsymbol{C}_{RR'}\, y^{R'} \Delta y^R. \tag{219}$$

For any displacement not at fixed τ the implication is that

$$\boldsymbol{\pi}_R(\tau, f)\, \{\Delta f^R - \partial f^R(\tau, y)/\partial \tau^S \,.\, \Delta \tau^S\} = \tfrac{1}{2} \boldsymbol{C}_{RR'}\, y^{R'} \Delta y^R. \tag{220}$$

Thus (218) becomes

$$\delta \int \{\tfrac{1}{2} \boldsymbol{C}_{RR'}\, y^{R'} \mathrm{D}y^R - H_S(\tau, y)\, \mathrm{D}\tau^S\}\, \mathrm{d}t = 0, \tag{221}$$

with

$$-H_S(\tau, y) = \boldsymbol{\pi}_S + \boldsymbol{\pi}_R\, \partial f^R(\tau, y)/\partial \tau^S. \tag{222}$$

Again, it is necessary to vary here only the y^R, though it yields no contradiction and no fresh information to vary $\mathscr{I}$ of the τ_S also. Finally, therefore, we have

THEOREM 8. *The extended motions* (112) *of any variational principle* (1), *regular or singular, admit regular canonical Lagrangian formulations* (221). *In such canonical formulations the canonical variables* y *are varied, while the gauge variables* τ *evolve freely. There is a complete unity of concept as between* t *and the gauge variables* τ.

Applications of the canonical equations

The canonical formalism has long been used as main support for the theory of contact transformations and Poisson brackets, and in effecting the transition to quantum mechanics. The work of the present section, with its complicated transformations and elaborate general formulae (215) and (222) for the Hamiltonian function, provides everything necessary to establish a complete link with these traditional methods, even for the most general time-dependent constraint-generating action principle (1). Thus it allows any new treatment to be matched against the old, and tested for equivalence. In particular, it can be used to prove that the global and coordinate-invariant methods of the third reference can be stated in a canonical manner. The canonical description gives manifest expression to the general properties of rank and nullity and determinism revealed by the work of the present paper. But as a practical calculational tool it is nevertheless cumbersome and difficult, and lacking in invariant emphasis. It is particularly awkward to formulate in those cases where $\mathbb{M}$ does not admit a globally single-

valued canonical covering. It is seldom the best framework for practical calculations. Practical calculations are most easily performed in the natural coordinates q^α of (1), in the manner set forth in the third reference. For this purpose the theorem of rank and nullity is essential, but canonical coordinates play no part at all.

I wish to thank Dr S. Brenner and Dr F. J. Bloore for several helpful discussions.

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